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Comparison of germ expansion on inner forms of $GL(n)$

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Abstract. We prove that the germ expansion of a discrete series representation π' on $GL_n(D)$ where D is a division algebra over k of index m and the germ expansion of the representation π of $GL_{mn}(k)$ associated to π' by the Deligne–Kazhdan–Vigneras correspondence are closely related, and therefore certain coefficients in the germ expansion of a discrete series representation of $GL_{mn}(k)$ can be interpreted (and therefore sometimes calculated) in terms of the dimension of a certain space of (degenerate) Whittaker models on $GL_n(D)$.

Let π be an irreducible admissible representation of $G = \underline{G}(k)$ for a reductive algebraic group $\underline{G}$ over a non-Archimedean local field k of characteristic 0. Then according to a theorem of Howe for GL_n, cf. [H], which was extended by Harish-Chandra, cf. [H-C], for general reductive groups, the character Θ_π of π can be written locally around the origin in G as

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi)\widehat{\mathcal{O}}(X),$$

where $\mathcal{O}$ runs over the set of nilpotent G-conjugacy classes in the Lie algebra $\mathfrak{g}$ of G, $\widehat{\mathcal{O}}$ denotes the Fourier transform of the distribution on the Lie algebra $\mathfrak{g}$ which is given by the integration on $\mathcal{O}$, and X belongs to $\mathfrak{g}$. It is known that $\widehat{\mathcal{O}}$ is a locally-L^1 function on $\mathfrak{g}$. The precise form of $c_{\mathcal{O}}(\pi)$ depends on fixing a Haar measure on $\mathfrak{g}$, an additive character $\psi : k \to \mathbb{C}^*$, and a choice of measure on the nilpotent orbit $\mathcal{O}$.

The calculation of the constants $c_{\mathcal{O}}(\pi)$ appearing in the germ expansion of the character of a representation π of the reductive group G is an important problem. The coefficient $c_{\mathcal{O}}(\pi)$ corresponding to the trivial nilpotent orbit is, up to a normalising factor, the formal degree of the representation π. The coefficient $c_{\mathcal{O}}(\pi)$ corresponding to maximal nilpotent orbits is, again up to a normalising factor, equal to the dimension of a space of Whittaker models according to theorems due to Rodier and Moeglin–Waldspurger, cf. [MW]. Besides these results, there are no other general theorems about the coefficients of the germ expansion. Calculations of these even in the context of $GL_3(k)$, $GL_4(k)$ seems to pose difficult

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problems, cf. [Mu]. In this note we observe that the germ expansion of a discrete series representation π' of $GL_n(D)$ where D is a division algebra over k of index m and the germ expansion of the representation π of $GL_{mn}(k)$ associated to π' by the Deligne–Kazhdan–Vigneras correspondence are closely related, and therefore certain coefficients in the germ expansion of a discrete series representation of $GL_{mn}(k)$ can be interpreted (and therefore sometimes calculated) in terms of the dimension of a certain space of (degenerate) Whittaker models on $GL_n(D)$. The point is that in the comparison of germ expansion of characters, coefficients $c_{\mathcal{O}'}(\pi')$ of maximal nilpotent orbit $\mathcal{O}'$ in $GL_n(D)$ corresponds to $c_{\mathcal{O}}(\pi)$ for an orbit $\mathcal{O}$ which is non-maximal (if D is not k). Therefore certain $c_{\mathcal{O}}(\pi)$ for non-maximal nilpotent orbits $\mathcal{O}$ in $GL_{mn}(k)$ get interpreted via the theorem of Moeglin–Waldspurger as the space of degenerate Whittaker models on $GL_n(D)$.

We recall that there is a bijective correspondence between conjugacy classes of nilpotent elements in $M_n(D)$ and the associated class of parabolic subgroups in $GL_n(D)$. (We recall that two parabolic subgroups are called associates if they have conjugate Levi subgroups.) This correspondence can be defined as follows. Let N be a nilpotent element in $M_n(D)$ operating on $V = D^n$. If we define $V_i = \ker(N^i)$, then $V_1 \subset V_2 \subset V_3 \cdots$. Define α_i to be the dimension of the D-vector space (V_{i+1}/V_i). Then $(\alpha_1 \geq \alpha_2 \geq \alpha_3 \cdots)$ is a partition of n and conversely every partition of n written in such a descending order arises from a unique conjugacy class of a nilpotent element. The parabolic P_N associated to the nilpotent element N consists of all elements in $GL_n(D)$ stabilizing the flag $V_1 \subset V_2 \subset V_3 \cdots$. This correspondence can be described geometrically using Richardson orbit theorem, but we will not do so here. We will denote the correspondence between conjugacy classes of nilpotent elements in $M_n(D)$ and conjugacy classes of parabolic subgroups in $GL_n(D)$ described here by $\mathcal{O} \to P_{\mathcal{O}}$. It is a result of Howe, cf. lemma 5 in [H], that the Fourier transform $\widehat{\mathcal{O}}$ of a nilpotent orbit $\mathcal{O}$ is, in a neighborhood of identity via the exponential map, a multiple of the character of the representation of $GL_n(D)$ on the space of locally constant functions on $GL_n(D)/P_{\mathcal{O}}$ on which $GL_n(D)$ acts by left translation. Howe actually treated only the case of D a field, but his proof works for $GL_n(D)$ too.

We next turn to the precise statement of the theorem of Deligne–Kazhdan–Vigneras, cf. [DKV].

Theorem 1. *Let D be a division algebra over a local field k of index m and dimension m^2 over k. Then there exists a bijective correspondence between discrete series representations π of $GL_{mn}(k)$ and discrete series representations π' of $GL_n(D)$ characterised by the character identity,*

$$(-1)^{mn}\Theta_\pi(x) = (-1)^n\Theta_{\pi'}(x)$$

where x is any regular conjugacy class shared by $GL_{mn}(k)$ and $GL_n(D)$.

We are now ready to state the main theorem of this paper. In this theorem we use the natural map from the set of nilpotent conjugacy classes in $M_n(D)$ to the set of nilpotent conjugacy classes in $M_{mn}(k)$ to identify nilpotent conjugacy classes in $M_n(D)$ to nilpotent conjugacy classes in $M_{mn}(k)$.

Theorem 2. *Let D be a division algebra over a local field k of index m and dimension m^2 over k. Let π be a discrete series representation of $GL_{mn}(k)$ and π' a discrete series representation of $GL_n(D)$ which correspond to each other by the Deligne–Kazhdan–Vigneras correspondence. Let $\mathcal{O}$ be a nilpotent orbit in the Lie algebra of $GL_{mn}(k)$ which corresponds to the nilpotent orbit $\mathcal{O}'$ in the Lie algebra of $GL_n(D)$. Choose a Haar measure on $M_{mn}(k)$, an additive character $\psi : k \to \mathbb{C}^*$, and a measure on the nilpotent orbit $\mathcal{O}$ so that the Fourier transform $\widehat{\mathcal{O}}$ of the nilpotent orbit $\mathcal{O}$ is, in a neighborhood of the identity, the character of the representation of $GL_{mn}(k)$ on the space of locally constant functions on $GL_{mn}(k)/P_{\mathcal{O}}$. Make a similar choice for $GL_n(D)$. Then one has the following equality of the coefficients in the germ expansion of π and π':*

$$(-1)^{mn} c_{\mathcal{O}}(\pi) = (-1)^n c_{\mathcal{O}'}(\pi').$$

Proof. From the character identity,

$$(-1)^{mn} \Theta_\pi(x) = (-1)^n \Theta_{\pi'}(x)$$

which is valid at all the regular semi-simple elements x shared by $GL_{mn}(k)$ and $GL_n(D)$, one has the following equality of germ expansions,

$$(-1)^{mn} \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi)\widehat{\mathcal{O}}(x) = (-1)^n \sum_{\mathcal{O}'} c_{\mathcal{O}'}(\pi')\widehat{\mathcal{O}'}(x).$$

By the theorem of Howe recalled earlier, we replace in the above sum, $\widehat{\mathcal{O}}$ by $\Theta_{GL_{mn}/P_{\mathcal{O}}}$, the character of the representation of $GL_{mn}(k)$ on the space of locally constant functions on $GL_{mn}(k)/P_{\mathcal{O}}$, and similarly for $GL_n(D)$. So,

$$(-1)^{mn} \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi)\Theta_{GL_{mn}/P_{\mathcal{O}}}(x) = (-1)^n \sum_{\mathcal{O}'} c_{\mathcal{O}'}(\pi')\Theta_{GL_n(D)/P_{\mathcal{O}'}}(x).$$

We now make the following observation whose simple proof we omit.

Observation. If a regular element X of $GL_{mn}(k)$ comes from $GL_n(D)$ and belongs to a parabolic P in $GL_{mn}(k)$, then the parabolic P comes from $GL_n(D)$, i.e., in the associated partition, $(\alpha_1 \geq \alpha_2 \geq \cdots)$, each α_i is divisible by m.

We now use the lemma according to which the character of the representation of a group G obtained on the space of functions on G/P is non-zero only for those conjugacy classes which intersect P. It follows from the observation above that the only characters $\Theta_{GL_{mn}/P_{\mathcal{O}}}$ which are non-zero at elements X coming from $GL_n(D)$ are those for which the associated parabolic $P_{\mathcal{O}}$ is one which makes sense for both $GL_{mn}(k)$ and $GL_n(D)$. Therefore one can write the above identity of characters as

$$(-1)^{mn} \sum_{\mathcal{O}'} c_{\mathcal{O}'}(\pi)\Theta_{GL_{mn}/P_{\mathcal{O}'}}(X) = (-1)^n \sum_{\mathcal{O}'} c_{\mathcal{O}'}(\pi')\Theta_{GL_n(D)/P_{\mathcal{O}'}}(X)$$

where now the sum on both the sides is over the conjugacy classes of nilpotent elements of $GL_n(D)$.

From the formula of van Dijk [vD], it is clear that for elements X whose conjugacy class lies in both $GL_{mn}(k)$ and $GL_n(D)$,

$$\Theta_{GL_{mn}/P_{\mathcal{O}'}}(X) = \Theta_{GL_n(D)/P_{\mathcal{O}'}}(X).$$

Therefore the identity of characters can be written as,

$$\sum_{\mathcal{O}'} \left[(-1)^{mn} c_{\mathcal{O}'}(\pi) - (-1)^n c_{\mathcal{O}'}(\pi')\right] \Theta_{GL_n(D)/P_{\mathcal{O}'}}(X) = 0.$$

The theorem now follows from the linear independence of $\widehat{\mathcal{O}'}$.

Definition 1. *Let ψ be a non-trivial additive character $\psi : D \to \mathbb{C}^*$. Using this, define a character Ψ on the group N of strictly upper triangular matrices in $GL_n(D)$ consisting of*

$$X = \begin{pmatrix} I & X_1 & * & * * & * \\ 0 & I & X_2 & * * & * \\ 0 & 0 & I & & \\ & & & \cdot & \\ & & & I & X_{n-1} \\ 0 & 0 & 0 & 0\,0 & I \end{pmatrix}$$

by $\Psi(X) = \psi(X_1 + \cdots + X_{n-1})$. For an irreducible admissible representation π of $GL_n(D)$, define $\pi_{N,\Psi}$ to be the largest quotient of π on which N operates via Ψ. The space $\pi_{N,\Psi}$ is called the space of degenerate Whittaker models of the representation π of $GL_n(D)$.

The following theorem is due to Moeglin and Waldspurger, cf. [MW].

Theorem 3. *For an irreducible admissible representation π of $GL_n(D)$, the space of degenerate Whittaker models of the representation π is finite dimensional and its dimension is equal to the leading term in the germ expansion of π.*

Corollary 1. *The coefficient of the germ expansion of a discrete series representation π of $GL_{mn}(k)$ corresponding to the nilpotent orbit containing the nilpotent element*

$$X = \begin{pmatrix} 0 & I & * & * & * & * \\ 0 & 0 & I & * & * & * \\ 0 & 0 & 0 & I & & \\ & & & \cdot & & \\ & & & & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where 0 and I in this matrix represent $m \times m$ trivial and identity matrices, is the same as the dimension of the space of degenerate Whittaker models of the representation π' of $GL_n(D)$ associated to the representation π by the Deligne–Kazhdan–Vigneras correspondence for D a division algebra of index m over k.

An explicit calculation

In this section we give an explicit formula for the dimension of the space of degenerate Whittaker models for a representation of $GL_2(D)$ from which the coefficient of the germ expansion for a supercuspidal representation of $GL_{2m}(k)$ at the unipotent element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ where 0 and 1 in this matrix represent $m \times m$ trivial and identity matrices, can be read. This coefficient of the germ expansion is also calculated by F. Murnaghan, cf. [Mu], in the case of $GL_4(k)$.

We begin by recalling that there are two maximal open compact-mod-center subgroups in $GL_2(D)$. To define them more precisely, let $\mathcal{O}_D$ denote the maximal compact ring of D, and $\mathfrak{P}$ a generator of the maximal ideal in $\mathcal{O}_D$. Let $K_1 = GL_2(\mathcal{O}_D)$ and let $K_2 = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL_2(\mathcal{O}_D) : C \equiv 0 \pmod{\mathfrak{P}}\}$. Let Z_1 be the cyclic group generated by $z_1 = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ and let Z_2 be the cyclic group generated by $z_2 = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$. Note that Z_1 normalizes K_1 and Z_2 normalizes K_2. Let $H_1 = Z_1K_1$ and let $H_2 = Z_2K_2$. Then H_1 and H_2 represent the two conjugacy classes of maximal open compact-mod-center subgroups of $GL_2(D)$.

Let H be one of the two maximal compact-mod-center subgroups as defined above. We define a class of representations of H, called *very cuspidal*, which when induced to $GL_2(D)$ produce irreducible supercuspidal representations. Both the definition and the proofs are exactly as in the field case. One would expect, though it is not yet known, that all supercuspidal representations of $GL_2(D)$ can be obtained in this way. Before we can define very cuspidal representations, we must first define a filtration on H.

Define a filtration on $M_2(D)$ indexed by $\mathbb{Z}$ as

$$A_1(d) := \mathfrak{P}^d M_2(\mathcal{O}_D).$$

A decreasing filtration $H_1(d)$ on H_1 is now defined by

$$H_1(d) := 1 + A_1(d)$$

for all $d \geq 1$, and $H_1(0) := GL_2(\mathcal{O}_D) = K_1$. Similarly define the filtration $A_2(d)$ on $M_2(D)$ as

$$A_2(d) := \mathfrak{P}^d \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{P} & \mathcal{O} \end{bmatrix}.$$

A decreasing filtration $H_2(d)$ is now defined by

$$H_2(d) := 1 + A_2(d)$$

for all $d \geq 1$; define $H_2(0) := K_2$.

Definition 2. *Let H be either H_1 or H_2. A finite dimensional irreducible representation (σ, W) of H is called* **very cuspidal of level d** *if :*

1. *It is trivial on $H(d)$.*
2. *It admits no fixed vector under the subgroup $N(\mathfrak{P}^{d-1}) \subset H/H(d)$, where $N(\mathfrak{P}^{d-1})$ is the group of upper-triangular unipotent matrices with coefficients in $\mathfrak{P}^{d-1}$.*

The following proposition summarizes some results proved in [PR]. This gives an explicit formula for the dimension of the space of degenerate Whittaker models for a representation π of $GL_2(D)$ in terms of formal degree and conductor of π, and hence by theorem 1 determines the coefficient of the (m, m) term in the germ expansion of a supercuspidal representation of $GL_{2m}(k)$.

Proposition 1. *Let (σ, W) be a very cuspidal representation of level d of H_i for $i = 1$ or 2. Then σ restricted to $N(\mathcal{O}_D)$, the group of upper-triangular unipotent matrices with coefficients in $\mathcal{O}_D$, breaks up into eigencharacters of $\mathcal{O}_D$ all of which have conductor $\mathfrak{P}^d$, i.e. trivial on $\mathfrak{P}^d$ and non-trivial on $\mathfrak{P}^{d-1}$. Further any such character occurs with the same multiplicity in σ. If we denote this common multiplicity as $r(\sigma)$ then $r(\sigma)$ is the dimension of the space of degenerate Whittaker models of the representation π of $GL_2(D)$ which is obtained by inducing σ from H_i to $GL_2(D)$. The conductor of the representation π is $2m + 2d + i - 3$. The dimension of σ, which is the same as the formal degree of π, is given by*

$$\dim(\sigma) = r(\sigma)q^{m(d-1)}(q^m - 1).$$

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