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KIRILLOV THEORY FOR $\mathrm{GL}_2(\mathcal{D})$ WHERE $\mathcal{D}$ IS A DIVISION ALGEBRA OVER A NON-ARCHIMEDEAN LOCAL FIELD

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Contents

1. Introduction and notation

1.1. Introduction. The aim of this work is to develop Kirillov theory for irreducible admissible representations of $\mathrm{GL}_2(\mathcal{D})$ for a division algebra $\mathcal{D}$ over a non-Archimedean local field F. We apply this theory to develop a theory of new forms for such representations.

The Kirillov theory developed here is in close analogy with the case of $\mathrm{GL}_2(F)$. We recall (see Jacquet and Langlands [12]) that the Kirillov model $K(\pi)$ of an irreducible admissible representation π of $\mathrm{GL}_2(F)$ consists of a certain space of locally constant functions on F^*, which vanish outside compact subsets of F, and contains $C_c^\infty(F^*)$ with codimension at most 2. The action of B, the standard Borel subgroup consisting of upper triangular matrices in $\mathrm{GL}_2(F)$, on $K(\pi)$ is given by

$$\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} f\right)(x) = \omega_\pi(d)\psi_F\left(d^{-1}xb\right) f\left(d^{-1}xa\right),$$

where ψ_F is a fixed nontrivial additive character of F, and ω_π is the central quasi character of π. The action of $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the Weyl group element, is given in terms of Fourier transforms. The explicit formula for this involves the ϵ-factors (actually γ-factors) of π twisted by characters of F^*.

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The theory of the Kirillov model depends on the existence and uniqueness of the Whittaker model for π, or equivalently on the existence and uniqueness of Whittaker functionals.

Definition 1.1. A linear functional $l : \pi \to \mathbb{C}$ is called a *Whittaker functional* if

$$l\left(\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix} v\right) = \psi_F(X) l(v)$$

for all v in π and for all X in F.

The basic theorem here is that every infinite-dimensional irreducible admissible representation π of $\mathrm{GL}_2(F)$ admits a nonzero Whittaker functional that is unique up to scalars.

For the representation theory of $\mathrm{GL}_2(\mathcal{D})$, we introduce a concept called *degenerate Whittaker functional*, which is defined as follows.

Definition 1.2. Let π be a representation of $\mathrm{GL}_2(\mathcal{D})$. A linear form $\ell : \pi \to \mathbb{C}$ is called a degenerate Whittaker functional if

$$\ell\left(\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix} v\right) = \Psi(X) \ell(v)$$

for all v in π and for all X in $\mathcal{D}$, where $\Psi(X) = \psi_F(T_{\mathcal{D}/F}(X))$. The reduced trace map from $\mathcal{D}$ to F is denoted by $T_{\mathcal{D}/F}$.

We need a concept called the *twisted Jacquet module* to analyze the space of degenerate Whittaker functionals.

Definition 1.3. Let (π, V) be a representation of $\mathrm{GL}_2(\mathcal{D})$. Define $V(N, \Psi)$ to be the $\mathbb{C}$-span of vectors of the form $\pi\left(\begin{smallmatrix}1 & X\\ 0 & 1\end{smallmatrix}\right)v - \Psi(X)v$ for $v \in V$ and for $X \in \mathcal{D}$. Define the twisted Jacquet module as $V_{N,\Psi} = V/V(N, \Psi)$.

Since $T_{\mathcal{D}/F}(ABA^{-1}) = T_{\mathcal{D}/F}(B)$, $V_{N,\Psi}$ is a module over

$$\Delta\mathcal{D}^* = \left\{\begin{pmatrix}A & 0\\ 0 & A\end{pmatrix} : A \in \mathcal{D}^*\right\}.$$

The twisted Jacquet module is the maximal quotient of V on which N acts via the character Ψ. The space of degenerate Whittaker functionals is just $\mathrm{Hom}_{\mathbb{C}}(V_{N,\Psi}, \mathbb{C})$.

By a theorem of Moeglin and Waldspurger [17], the twisted Jacquet module of any irreducible admissible representation is finite-dimensional. We refer to the second author's thesis [23] for details on this.

In Section 2 the twisted Jacquet module is explicitly computed for parabolically induced representations from which finite dimensionality falls out (see Theorem 2.1).

In Section 3 we begin our analysis of Kirillov theory. The Kirillov model that we consider is realized on a space of functions on $\mathfrak{D}^*$ with values in $V_{N,\Psi}$. We denote this space of functions by $\mathcal{K}(\pi)$. It turns out that $\mathcal{K}(\pi)$ contains $C_c^\infty(\mathfrak{D}^*, V_{N,\Psi})$ as a subspace with finite codimension and the minimal parabolic subgroup P consisting of upper triangular matrices acts on $\mathcal{K}(\pi)$ by the formula

$$\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} f\right)(X) = \psi\big(D^{-1}XB\big)\pi\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} f\big(D^{-1}XA\big). \tag{1}$$

Furthermore, the representation π is supercuspidal if and only if $\mathcal{K}(\pi) = C_c^\infty(\mathfrak{D}^*, V_{N,\Psi})$. The main theorem of this section is recorded in Theorem 3.1. As a corollary to this theorem, we get that taking duals commutes with taking the twisted Jacquet module (see Proposition 3.1).

To have a complete picture as to how G acts on the Kirillov space, we need to describe how the Weyl group element acts on $\mathcal{K}(\pi)$. Since $w\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)w^{-1} = \left(\begin{smallmatrix} D & 0 \\ 0 & A \end{smallmatrix}\right)$, w gives rise to an intertwining on the Kirillov model $\mathcal{K}(\pi)$ for the corresponding two actions of $\mathfrak{D}^* \times \mathfrak{D}^*$. It should be possible to make this action more explicit, but we have not been able to do that.

We apply Kirillov theory as developed in Section 3 to develop a theory of new forms considered in Section 4. The statements are akin to Casselman [7], although our proofs are modeled on Deligne [8]. Given an irreducible admissible representation (π, V) of G, we consider vectors fixed under a certain congruence subgroup $\Gamma_0^1(m)$. Let V_m denote $V^{\Gamma_0^1(m)}$, the space of vectors in V that are fixed by $\Gamma_0^1(m)$. We prove in Section 4 that for m large enough, there are vectors fixed under $\Gamma_0^1(m)$, and the space of fixed vectors is intimately connected with the twisted Jacquet module of π. The main theorem proved about new forms is that if $C(\pi)$ (called the conductor of π in the sense of new forms) is the smallest nonnegative integer m for which $V_m \neq (0)$, then $V_{C(\pi)}$ is isomorphic to $V_{N,\Psi}$ as $\mathfrak{D}^*$-modules. (Actually, we are able to prove this only for irreducible principal series and those supercuspidal representations that are obtained by compact induction.) Every time we increase the level by 1, one more copy of $V_{N,\Psi}$ gets added to the space of fixed vectors. We also derive an explicit formula connecting the conductor of π in the sense of new forms and the exponent of the epsilon factor attached to π as in [10]. These results are considered in Section 4.1 for principal series, and Section 5 for supercuspidal representations (see Theorems 4.1, 5.1, 5.2, and 5.3). In Section 4.2, we also give an explicit form of the spherical vector in the Kirillov model for unramified principal series representations of $GL_2(\mathfrak{D})$ (see Theorem 4.2).

In Section 5 we construct a family of supercuspidal representations of G. This is done by identifying what are called very cuspidal representations of maximal open compact mod center subgroups of G. Compactly inducing them to G gives supercuspidal representations. We then take up these representations and compute their twisted Jacquet modules, conductors in the sense of epsilon factors and also in the sense of new forms, and finally identify the space of new forms as the twisted Jacquet module (see Propositions 5.2, 5.3, and 5.4).

In Section 6 we take up the concept of the *Shalika model* (see Definition 6.1), which is closely related to the Kirillov model. We prove that if π admits a Shalika model, then it admits a unique one. We also prove that if π admits a Shalika model, then π is self-contragredient (see Theorem 6.2).

The present work highlights the importance of the space of degenerate Whittaker models for the representation theory of $\mathrm{GL}_2(\mathcal{D})$. It would be very nice if there was a way of predicting the structure of the space of degenerate Whittaker model as a representation space of $\mathcal{D}^*$. A conjecture of B. H. Gross and the first author gives an answer to this question in terms of certain local root numbers when $\mathcal{D}$ is a quaternion division algebra (which is the only case when representations of $\mathcal{D}^*$ occurring in the space of degenerate Whittaker models appear with multiplicity 1). This conjecture and some of its consequences are elaborated upon in [22].

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1.2. Notation. Let F be a non-Archimedean local field. Let $\mathcal{D}$ be a division algebra with center F and of index d over F, that is, of dimension d^2 over F. Let $\mathcal{O}_F$ be the ring of integers in F, and let $\mathcal{O}$ be the ring of integers in $\mathcal{D}$. Let ϖ_F be a uniformizer in F, and let ϖ be a uniformizer in D such that $\varpi^d = \varpi_F$. Let $\mathfrak{P}_F$ be the unique maximal ideal in $\mathcal{O}_F$, and let $\mathfrak{P}$ be the unique maximal ideal in $\mathcal{O}$. Let $\mathfrak{v}_F$ be the valuation on F with $\mathfrak{v}_F(\varpi_F) = 1$, and let $\mathfrak{v}_{\mathcal{D}}$ be the valuation on $\mathcal{D}$ with $\mathfrak{v}_{\mathcal{D}}(\varpi) = 1$. These valuations uniquely determine the normalized multiplicative valuations $|\cdot|_F$ and $|\cdot|$ on F and $\mathcal{D}$, respectively, by the formulae $|\varpi_F|_F = q^{-1}$ and $|\varpi| = q^{-d}$, where q is the cardinality of the residue field of F. Let $T_{\mathcal{D}/F}$ be the reduced trace map from $\mathcal{D}$ to F. Let $N_{\mathcal{D}/F}$ denote the reduced norm map from $\mathcal{D}$ to F. Let ψ_F be a nontrivial additive character on F so chosen that the maximal fractional ideal in F on which ψ_F is trivial is $\mathcal{O}_F$. Let Ψ be the character on $\mathcal{D}$ obtained by composing the reduced trace and the character ψ_F.

Let $\mathcal{M} = M_2(\mathcal{D})$ be the matrix algebra of 2×2 matrices with entries in $\mathcal{D}$. It is a central simple algebra over F of dimension $4d^2$. Let G stand for the group $\mathrm{GL}_2(\mathcal{D}) = \mathcal{M}^\times$. So G may be regarded as the F-points of a linear algebraic group defined over F. Let P be the minimal parabolic subgroup of upper triangular matrices in G. Let $P = M \cdot N$, where M is the Levi part of P consisting of diagonal matrices and N is the unipotent radical of P consisting of upper triangular matrices with 1's on the diagonal. Let $\overline{P}$ denote the parabolic subgroup opposed to P consisting of lower triangular matrices. The character Ψ of $\mathcal{D}$ is also thought of as a character of N. Let S be the "Shalika subgroup" of P consisting of all matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix}\right)$ in P. Then S is the subgroup of P consisting of all elements that leave Ψ invariant.

Let ΔD^* be $M \cap S$. Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be the Weyl group element in G. Let K denote the maximal open compact subgroup $\mathrm{GL}_2(\mathcal{O})$.

Let H be an l-group (see [1] for the definition of an l-group), and let B be a closed subgroup of H. Let (σ, W) be a smooth representation of B. The notation $\mathrm{Ind}_B^H(\sigma)$ stands for unnormalized induction from B to H of the representation σ of B. The notation $\mathrm{ind}_B^H(\sigma)$ stands for the unnormalized compact induction from B to H of the representation σ of B.

1.3. Basic structure theory of $\mathrm{GL}_2(\mathcal{D})$. In this section we collect some theorems on the structure of $\mathrm{GL}_2(\mathcal{D})$, which are used in the rest of this article. No proofs are given as these are all well known and easy to prove.

PROPOSITION 1.1 (Bruhat decomposition). *With the notation as above, we have*

$$G = P \amalg PwP.$$

Note that the "big Bruhat cell" PwP can also be written as $NwMN = NMwN$.

PROPOSITION 1.2 (Iwasawa decomposition). *With the notation as above, we have*

$$G = K \cdot P = P \cdot K.$$

PROPOSITION 1.3 (Cartan decomposition). *With the notation as above, we have*

$$G = K \cdot A \cdot K,$$

where A is the submonoid of G generated by the elements $\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix}$.

PROPOSITION 1.4 (Maximal open compact mod center subgroups). *Let $K_1 = \mathrm{GL}_2(\mathcal{O})$, and let $K_2 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}) : C \equiv 0 \pmod{\mathfrak{P}} \right\}$. Let Z_1 be the cyclic group generated by $z_1 = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix}$, and let Z_2 be the cyclic group generated by $z_2 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$. Note that Z_1 normalizes K_1, and Z_2 normalizes K_2. Let $H_1 = Z_1 K_1$, and let $H_2 = Z_2 K_2$. Then H_1 and H_2 are maximal, open, compact modulo center subgroups of G, and any subgroup that is open and compact modulo center can be conjugated inside H_1 or H_2. Furthermore, H_1 and H_2 are not conjugate to each other.*

For a proof of Proposition 1.4, refer to [4].

PROPOSITION 1.5 (Iwasawa decomposition II). *With the notation as above, we have*

(1) $G = H_1 \cdot P = P \cdot H_1$,
(2) $G = H_2 \cdot P = P \cdot H_2$.

PROPOSITION 1.6 (Cartan decomposition II). *With the notation as above, we have*

(1) $G = H_1 \cdot A \cdot H_1$,
(2) $G = H_2 \cdot A \cdot H_2$,
(3) $G = H_2 \cdot A \cdot H_1$.

We need to consider certain congruence subgroups of $K = \mathrm{GL}_2(\mathbb{O})$. Let

$$\Gamma_0(m) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathbb{O}) : C \equiv 0 \, \left(\mathrm{mod}\, \mathfrak{P}^m\right) \right\}$$

and

$$\Gamma_0^1(m) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathbb{O}) : C \equiv 0,\ D \equiv 1 \, \left(\mathrm{mod}\, \mathfrak{P}^m\right) \right\}.$$

We also use the following notation. If $X_1, \dots, X_4$ are subsets of $\mathfrak{D}$, then

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} := \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} : A_i \in X_i \right\}.$$

PROPOSITION 1.7 (Iwahori factorization). *With the notation as above, we have*
(1) $\Gamma_0(m) = \overline{N}(\mathfrak{P}^m)\left[\begin{smallmatrix} \mathbb{O}^\times & 0 \\ 0 & \mathbb{O}^\times \end{smallmatrix}\right] N(\mathbb{O})$,
(2) $\Gamma_0^1(m) = \overline{N}(\mathfrak{P}^m)\left[\begin{smallmatrix} \mathbb{O}^\times & 0 \\ 0 & 1+\mathfrak{P}^m \end{smallmatrix}\right] N(\mathbb{O})$.
Here $\overline{N}(\mathfrak{P}^m) = \overline{N}(m) = \overline{N} \cap \Gamma_0(m)$ *and* $N(\mathbb{O}) = N \cap K$.

PROPOSITION 1.8. *For all* $m \geq 1$, *the subgroup of* G *generated by* $\Gamma_0^1(m)$ *and* $\overline{N}(m-1)$ *is* $\Gamma_0^1(m-1)$.

PROPOSITION 1.9. *The reduced trace map* $T_{\mathfrak{D}/F}$ *has the property*

$$T_{\mathfrak{D}/F}(\mathfrak{P}^m) = \mathfrak{P}_F^{[(m+d-1)/d]}.$$

The conductor of the character Ψ *of* $\mathfrak{D}$ *is* $\mathfrak{P}^{1-d}$.

2. Jacquet modules of principal series representations. In this section we explicitly calculate the Jacquet module and the twisted Jacquet module for a parabolically induced representation of G.

Let π_1 and π_2 be irreducible (necessarily finite-dimensional) representations of $\mathfrak{D}^*$. Consider the representation $\pi_1 \otimes \pi_2 \otimes \Delta^{1/2}$ of M where Δ is the character of $\mathfrak{D}^* \times \mathfrak{D}^*$ defined by $\Delta(X, Y) = |XY^{-1}|$. We think of this as a representation of P by extending it trivially across N. Denote by $V(\pi_1, \pi_2)$ the representation of G induced from this representation of P. So $V(\pi_1, \pi_2)$ is the space of functions

$$\left\{ f \in C^\infty\big(G, \pi_1 \otimes \pi_2\big) : f\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} g\right) = \Delta(A, D)^{1/2}\big(\pi_1(A) \otimes \pi_2(D)\big) f(g) \right\}.$$

We call $V(\pi_1, \pi_2)$ a *principal series representation.* In general it is not irreducible, and when it is irreducible, we explicitly mention it.

The following theorem computes the twisted Jacquet module for such a representation $V(\pi_1, \pi_2)$.

THEOREM 2.1. *For irreducible representations π_1 and π_2 of $\mathcal{D}^*$, let $V(\pi_1, \pi_2)$ be the representation of $\mathrm{GL}_2(\mathcal{D})$ as defined above. Then there is a natural isomorphism of $\mathcal{D}^*$-modules $(V(\pi_1, \pi_2))_{N,\Psi} \simeq \pi_1 \otimes \pi_2$.*

Proof. The proof is a simple consequence of the Bruhat decomposition, which gives rise to the following exact sequence of P-modules:

$$0 \longrightarrow C_c^\infty(N) \otimes \pi_1 \otimes \pi_2 \longrightarrow V(\pi_1, \pi_2) \longrightarrow \pi_1 \otimes \pi_2 \otimes \Delta^{1/2} \longrightarrow 0$$

and the elementary fact that the twisted Jacquet functor of $C_c^\infty(N)$ is $\mathbb{C}$. We refer to [21, Proposition 7] for more details. □

The Jacquet module of $V(\pi_1, \pi_2)$, namely, the maximal quotient of $V(\pi_1, \pi_2)$ on which N acts trivially, can also be computed as in the proof of the above theorem. We state this as the following theorem and leave the reader to fill in the details.

THEOREM 2.2. *For irreducible representations π_1 and π_2 of $\mathcal{D}^*$, let $V(\pi_1, \pi_2)$ denote the corresponding principal series representation of G. The semisimplification of the Jacquet module of $V(\pi_1, \pi_2)$ is given by $\Delta^{1/2} \cdot [(\pi_1 \otimes \pi_2) \oplus (\pi_2 \otimes \pi_1)]$ as M-modules.*

3. Kirillov theory. In this section we develop Kirillov theory for irreducible representations of G. We prove that an irreducible admissible representation (π, V) of $\mathrm{GL}_2(\mathcal{D})$ can be realized on a certain space of functions on $\mathcal{D}^*$ with values in a finite-dimensional vector space, namely, $V_{N,\Psi}$, on which the parabolic subgroup P acts in a very explicit way. This space of functions contains $C_c^\infty(\mathcal{D}^*, V_{N,\Psi})$ as a subspace of finite codimension. By Proposition 1.1, we just need to know how the Weyl group element acts to get a complete understanding of π, but which we have not been able to achieve here.

Let $L : V \to V_{N,\Psi} = V/V(N, \Psi)$ be the canonical projection. For any $v \in V$, let φ_v denote the $V_{N,\Psi}$ valued function on $\mathcal{D}^*$ given by

$$\varphi_v(A) = L\left(\pi \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} v\right).$$

We state some preliminary lemmas that give some properties of the functions φ_v's on $\mathcal{D}^*$ with values in $V_{N,\Psi}$. These lemmas are almost identical to the case of $\mathrm{GL}_2(F)$ as written in [9]. We omit the proofs of these lemmas, as the proofs given for the corresponding statements in [9] go through mutatis mutandis to our case.

LEMMA 3.1. *If $v' = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) v$, then*

$$\varphi_{v'}(X) = \Psi\big(D^{-1}XB\big)\pi \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \varphi_v\big(D^{-1}XA\big).$$

LEMMA 3.2. *The function* φ_v *is a locally constant function on* $\mathcal{D}^*$ *that vanishes outside a compact subset of* $\mathcal{D}$.

LEMMA 3.3. *The map* $v \mapsto \varphi_v$ *from* V *to* $C^\infty(\mathcal{D}^*, V_{N,\Psi})$ *is an injective linear map.*

Let $\mathcal{K}(\pi)$ denote $\{\varphi_v : v \in V\}$. From the previous lemmas, $\mathcal{K}(\pi)$ is a $\mathbb{C}$-vector space of locally constant functions on $\mathcal{D}^*$ with values in $V_{N,\Psi}$, vanishing outside compact subsets of $\mathcal{D}$, on which P acts by a simple formula. By Lemma 3.3, since the map $v \mapsto \varphi_v$ is injective, we can give a G action to $\mathcal{K}(\pi)$ by borrowing the G action on V.

The map $v \mapsto \varphi_v$ has a fairly natural interpretation as follows. If (ρ, W) is any representation of S, then by Frobenius reciprocity, we have

$$\mathrm{Hom}_P\left(\pi, \mathrm{Ind}_S^P(\rho)\right) \simeq \mathrm{Hom}_S(\pi|_S, \rho).$$

If N acts via Ψ on ρ, then since $V_{N,\Psi}$ is the largest quotient of V on which N acts via Ψ, we have

$$\mathrm{Hom}_S(\pi|_S, \rho) \simeq \mathrm{Hom}_S(\pi_{N,\Psi}, \rho).$$

Taking ρ as $\pi_{N,\Psi}$, we get

$$\mathrm{Hom}_P\left(\pi, \mathrm{Ind}_S^P(\pi_{N,\Psi})\right) \simeq \mathrm{Hom}_S(\pi_{N,\Psi}, \pi_{N,\Psi}).$$

It is easy to see that the map $v \mapsto \varphi_v$ is the pullback Φ of the identity map on $V_{N,\Psi}$ in the above isomorphism. We have implicitly identified functions in $\mathrm{Ind}_S^P(\rho)$ as functions on $\mathcal{D}^*$ with values in ρ on which P acts exactly as in Lemma 3.1. So we have $\Phi(v) = \varphi_v$ for all $v \in V$, and the image of Φ is $\mathcal{K}(\pi)$ and, in particular, $\Phi(v)(1) = L(v)$.

The next lemma can be thought of as a p-adic analogue of the method of little groups of Mackey and Wigner (see [24, Section 8.2]) of describing irreducible representations of a group that is an extension of some group by an abelian group.

LEMMA 3.4. *Given an irreducible representation* (σ, W) *of* S*, the Shalika subgroup of* G*, on which* N *acts via* Ψ*, the representation* $\mathrm{ind}_S^P(\sigma)$ *of* P*, obtained by compactly inducing* σ *to* P*, is irreducible.*

Proof. The proof is based on the proof of [3, Proposition 4.7.3]. We can consider the representation space of $\mathrm{ind}_S^P(\sigma)$ as $C_c^\infty(\mathcal{D}^*, W)$ on which P acts via the formula

$$\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} f\right)(X) = \Psi\big(D^{-1}XB\big)\sigma\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} f\big(D^{-1}XA\big).$$

If $v \in W$ and U is an open compact set in $\mathcal{D}^*$, let $f_{U,v}$ be the function that takes the value v on U and 0 elsewhere. Clearly, such functions $f_{U,v}$ span $V = C_c^\infty(\mathcal{D}^*, W)$ as v and U vary.

Let V_1 be a nonzero, P-stable subspace of V, and let

$$W_1 = \{f(X) : X \in \mathfrak{D}^*,\ f \in V_1\}.$$

Let $0 \neq f \in V_1$. Therefore, there exists $A \in \mathfrak{D}^*$ such that $f(A) = v \neq 0$. For $\phi \in C_c^\infty(\mathfrak{D})$, let

$$f_\phi = \int_{\mathfrak{D}} \phi(X)\left(\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} f\right) dX,$$

which gives that

$$f_\phi(Y) = \int_{\mathfrak{D}} \phi(X)\Psi(XY) f(Y)\, dX = \widehat{\phi}(Y) f(Y).$$

Note that f_ϕ is in V_1. Now choose ϕ such that $\widehat{\phi}$ is the characteristic function of U, an open compact subset of $\mathfrak{D}^*$ containing A on which f is constant, and hence the constant value is v, to get that $f_\phi = f_{U,v}$.

The formula

$$\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} f_{U,v} = f_{DUD^{-1},\sigma\left(\begin{smallmatrix} D & 0 \\ 0 & D \end{smallmatrix}\right)v}$$

implies that $W_1 = W$.

So now choose any $w \in W$ and choose a function $g \in V_1$ such that $g(C) = w$ for some $C \in \mathfrak{D}^*$. Looking at g_ϕ and choosing ϕ appropriately, we get that for arbitrarily small neighbourhoods U of C, $f_{U,w} \in V_1$. The formula

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} f_{U,w} = f_{U\cdot A^{-1},w}$$

implies that given any $w \in W$ and any $A \in \mathfrak{D}^*$ for all small enough neighbourhoods U of A, we have $f_{U,w} \in V_1$. Hence, $V_1 = V$, which is what we wanted to prove. □

THEOREM 3.1. (1) *For all $n \in N$ and for all $v \in V$, $n \cdot \varphi_v - \varphi_v$ has compact support in $\mathfrak{D}^*$.*

(2) *The Kirillov space $\mathcal{K}(\pi)$ contains all functions in $C_c^\infty(\mathfrak{D}^*, V_{N,\Psi})$.*

(3) *The Jacquet module of π, namely, the maximal quotient of π on which N acts trivially, denoted by π_N, is $\mathcal{K}(\pi)/C_c^\infty(\mathfrak{D}^*, V_{N,\Psi})$.*

(4) *The representation π is supercuspidal if and only if $C_c^\infty(\mathfrak{D}^*, V_{N,\Psi}) = \mathcal{K}(\pi)$.*

(5) *The Jacquet module is finite-dimensional, that is, $C_c^\infty(\mathfrak{D}^*, V_{N,\Psi})$ has finite codimension in $\mathcal{K}(\pi)$.*

Proof. Let $f = n \cdot \varphi_v - \varphi_v$, where $n = \left(\begin{smallmatrix} 1 & A \\ 0 & 1 \end{smallmatrix}\right)$. Then $f(X) = (\Psi(XA) - 1)\varphi_v(X)$. So for all $X \in A^{-1}\mathcal{O}$, we get $f(X) = 0$ since the conductor of Ψ is $\mathfrak{P}^{1-d}$. By Lemma 3.2, the support of the function f lies in a compact subset of $\mathfrak{D}$, and by the above argument, this compact subset does not contain 0; hence, it is a compact subset of $\mathfrak{D}^*$. This proves (1).

Note that $C_c^\infty(\mathcal{D}^*, V_{N,\Psi}) = \mathrm{ind}_S^P(V_{N,\Psi})$. The twisted Jacquet module $V_{N,\Psi}$ is a finite-dimensional $S = \Delta\mathcal{D}^* \cdot N$-module on which N acts via Ψ. Since $\Delta\mathcal{D}^*$ is compact modulo center and the center acts via ω_π, the central character of π, $V_{N,\Psi}$ is a semisimple module over S. So let $V_{N,\Psi} = m_1\pi_1 \oplus m_2\pi_2 \oplus \cdots \oplus m_k\pi_k$, where π_i's are mutually inequivalent irreducible S-modules. Induction being an exact functor, we have

$$\mathrm{Ind}_S^P(V_{N,\Psi}) = \oplus_{i=1}^k m_i \,\mathrm{Ind}_S^P(\pi_i),$$

$$\mathrm{ind}_S^P(V_{N,\Psi}) = \oplus_{i=1}^k m_i \,\mathrm{ind}_S^P(\pi_i).$$

For any nonzero vector $\alpha \in \pi_i$ and any $1 \le j \le m_i$, let v be a vector in V such that $L(v)$ is α in the jth copy of π_i and 0 everywhere else. (Here L is the canonical map from V to $V_{N,\Psi}$). Let f be the corresponding function in $\mathcal{K}(\pi)$. Let $f = \sum f_r^s$ with $f_r^s \in \mathrm{Ind}_S^P(\pi_r)$ as the sth copy. So $f_r^s(1) = 0$ unless $r = i$ and $s = j$, and $f_i^j(1) = \alpha$. Take any $n \notin N(\mathfrak{P}^{1-d})$ such that $\Psi(n) \neq 1$, and let $g = n \cdot f - f$. Then by (1), g is in $C_c^\infty(\mathcal{D}^*, V_{N,\Psi})$. If we write g as $\sum g_r^s$, then $g_r^s(1) = 0$ unless $r = i$ and $s = j$, in which case it is $(\Psi(n) - 1)\alpha \neq 0$. As in the proof of the previous lemma, take any $\phi \in C_c^\infty(\mathcal{D})$, and let $g_\phi = \int_{\mathcal{D}} \phi(X) \left(\begin{pmatrix} 1 & X \\ 0 & 1\end{pmatrix} g\right) dX$, which is essentially a finite sum as g has compact support in $\mathcal{D}^*$, and so we get $g_\phi(Y) = \widehat{\phi}(Y) g(Y) = \sum \widehat{\phi} g_r^s$. Now choose ϕ such that $\widehat{\phi}$ is the characteristic function of U, an open compact neighbourhood of 1, such that all the functions g_r^s are constant on U. Therefore, g_ϕ is a function that takes the constant value α on U and is 0 elsewhere. Hence, $g_\phi \in \mathrm{ind}_S^P(\pi_i^j)$, where π_i^j is π_i sitting as the jth copy in $V_{N,\Psi}$. So for all $1 \le i \le k$ and all $1 \le j \le m_i$, $\mathcal{K}(\pi) \cap \mathrm{ind}_S^P(\pi_i^j)$ is nonempty and so by Lemma 3.4, $\mathrm{ind}_S^P(\pi_i^j) \subset \mathcal{K}(\pi)$. Hence, $\mathrm{ind}_S^P(V_{N,\Psi}) \subset \mathcal{K}(\pi)$. This proves (2).

To prove (3) using (1), we just need to show that any $f \in C_c^\infty(\mathcal{D}^*, V_{N,\Psi})$ is a finite sum of functions as in (1). So for any such f, consider

$$g = \int_{\mathfrak{P}^{-m}} \pi \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \cdot f \, dX.$$

So for all A in $\mathcal{D}^*$, we have

$$g(A) = \int_{\mathfrak{P}^{-m}} \left(\pi \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \cdot f\right)(A)\, dX = \int_{\mathfrak{P}^{-m}} \Psi(AX) f(A)\, dX.$$

Since f has compact support, there exists $a \le b$ such that $\mathrm{supp}(f) \subset \mathfrak{P}^a$ and $f(\mathfrak{P}^b) = 0$. Choose any $m \ge b + d$. Then $X \mapsto \Psi(AX)$ is a nontrivial character on $\mathfrak{P}^{-m}$ for any $A \in \mathrm{supp}(f)$. This gives that g is identically zero, from which it follows that f is a finite sum of functions of the form $\pi(n)\phi - \phi$. This proves (3).

Statement (4) follows from (3) and the definition of supercuspidality. Statement (5) follows from (4) if π is a supercuspidal representation, and if π is a subrepresentation of a parabolically induced representation, then it follows from Theorem 2.2. □

PROPOSITION 3.1. *If* (π, V) *is an irreducible admissible representation of* G, *let* $(\pi^\vee, V^\vee)$ *be its contragredient representation. Then there is a natural isomorphism of* $\mathfrak{D}^*$ *modules*

$$(\pi_{N,\Psi})^\vee \simeq (\pi^\vee)_{N,\overline{\Psi}}.$$

Proof. By Theorem 3.1(3), we have the exact sequence of P-modules

$$0 \longrightarrow \mathrm{ind}_S^P(\pi_{N,\Psi}) \longrightarrow \pi \longrightarrow \pi_N \longrightarrow 0,$$

where π_N is the usual Jacquet module of π. Let $(\pi|_P)^\vee$ be the smooth dual of π considered as a P-module. So $\pi^\vee \subset (\pi|_P)^\vee$. Dualizing the above exact sequence of P-modules, we get

$$0 \longrightarrow (\pi_N)^\vee \longrightarrow (\pi|_P)^\vee \longrightarrow \mathrm{Ind}_S^P\left((\pi_{N,\Psi})^\vee\right) \longrightarrow 0.$$

Since N acts trivially on π_N, $((\pi_N)^\vee)_{N,\overline{\Psi}} = (0)$. Therefore,

$$\left((\pi|_P)^\vee\right)_{N,\overline{\Psi}} \simeq \mathrm{Ind}_S^P\left((\pi_{N,\Psi})^\vee\right)_{N,\overline{\Psi}}.$$

If (σ, U) is a finite-dimensional representation of the Shalika subgroup S on which N acts via a fixed nontrivial character $\overline{\Psi}$, then identifying $\mathrm{Ind}_S^P(\sigma)$ with functions on $\mathfrak{D}^*$ with values in U, we get exactly as in statements (1) and (3) of Theorem 3.1 that N operates trivially on $\mathrm{Ind}_S^P(\sigma)/\mathrm{ind}_S^P(\sigma)$. Hence, $(\mathrm{Ind}_S^P(\sigma))_{N,\overline{\Psi}} \simeq (\mathrm{ind}_S^P(\sigma))_{N,\overline{\Psi}} \simeq \sigma$. Taking $\sigma = (\pi_{N,\Psi})^\vee$, we find that

$$\left((\pi|_P)^\vee\right)_{N,\overline{\Psi}} \simeq (\pi_{N,\Psi})^\vee.$$

Thus, under the natural inclusion of $\pi^\vee$ into $(\pi|_P)^\vee$, $(\pi^\vee)_{N,\overline{\Psi}}$ becomes a submodule of $(\pi_{N,\Psi})^\vee$. Interchanging the roles of π and $\pi^\vee$ and noting that $(\pi^\vee)_{N,\overline{\Psi}}$ is finite-dimensional completes the proof of the proposition. □

We omit the proof of the following easy corollary in which we give a P-equivariant pairing between an irreducible supercuspidal representation of G and its contragredient in terms of their Kirillov models.

COROLLARY 3.1. *Let* π *be an irreducible admissible supercuspidal representation of* $G = \mathrm{GL}_2(\mathfrak{D})$. *Let* $\mathcal{K}_\Psi(\pi)$ *and* $\mathcal{K}_{\overline{\Psi}}(\pi^\vee)$ *be the Kirillov models of* π *and its contragredient* $\pi^\vee$ *with respect to the additive characters* Ψ *and* $\overline{\Psi}$, *respectively. Using the identification in Proposition 3.1, the map* $\mathcal{K}_\Psi(\pi) \times \mathcal{K}_{\overline{\Psi}}(\pi^\vee) \to \mathbb{C}$ *given by sending the pair* (f, g) *for* $f \in \mathcal{K}_\Psi(\pi)$ *and* $g \in \mathcal{K}_{\overline{\Psi}}(\pi^\vee)$ *to the number* $\langle f, g\rangle$ *defined by*

$$\langle f, g\rangle = \int_{\mathfrak{D}^*} \langle f(X), g(X)\rangle \, dX,$$

gives a P*-invariant duality between* $\mathcal{K}_\Psi(\pi)$ *and* $\mathcal{K}_{\overline{\Psi}}(\pi^\vee)$.

Remark 3.1. If $\mathfrak{D}$ is a field, we know that an irreducible supercuspidal representation of $\mathrm{GL}_2(\mathfrak{D})$ remains irreducible when restricted to P (see [1, Section 5]). It follows that the natural $\mathrm{GL}_2(\mathfrak{D})$-invariant pairing on $\pi \times \pi^\vee$ is the unique P-invariant bilinear pairing on $\pi \times \pi^\vee$. Therefore, the pairing in Corollary 3.1 is automatically $\mathrm{GL}_2(\mathfrak{D})$-invariant. When $\mathfrak{D}$ is not a field, an irreducible supercuspidal representation of $\mathrm{GL}_2(\mathfrak{D})$ may not be irreducible when restricted to P (equivalently, the space of degenerate Whittaker models may not be irreducible as a $\mathfrak{D}^*$-module), and therefore it is not clear if the pairing defined in Corollary 3.1 is $\mathrm{GL}_2(\mathfrak{D})$ invariant in general.

4. New forms. In this section we investigate the space of fixed vectors under a certain type of congruence subgroup for any irreducible admissible representation of G. Define for $m \geq 1$,

$$\Gamma_0^1(m) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K : C \equiv 0,\ D \equiv 1 \left(\operatorname{mod} \mathfrak{P}^m \right) \right\}$$

and $\Gamma_0^1(0) = K = \mathrm{GL}_2(\mathcal{O})$. By a *new form* for (π, V) we mean a vector in V fixed under $\Gamma_0^1(m)$ for some m but that is not fixed under $\Gamma_0^1(m-1)$ if $m \geq 1$, or a nonzero vector fixed under $\Gamma_0^1(0)$. For all $m \geq 0$, we use the notation

$$V_m := V^{\Gamma_0^1(m)}.$$

Also, for convenience, let $V_{-1} := (0)$.

We start with a proposition that states that new forms exist.

PROPOSITION 4.1. *If (π, V) is an irreducible admissible infinite-dimensional representation of G, then there exists an integer $m \geq 0$ such that π admits a $\Gamma_0^1(m)$-fixed vector.*

Proof. Since V is infinite-dimensional, $V_{N,\Psi}$ is not (0). Let α be any vector in $V_{N,\Psi}$, and let f_α be the function that is 0 outside $\mathcal{O}^\times$ and takes the value α on $\mathcal{O}^\times$. From Theorem 3.1, since f_α is in $C_c^\infty(\mathfrak{D}^*, V_{N,\Psi})$, it belongs to $\mathcal{K}(\pi)$. For all $A \in \mathcal{O}^\times$ and for all $B \in \mathcal{O}$, it is easily checked that f_α is left-invariant by $\left(\begin{smallmatrix} A & B \\ 0 & 1 \end{smallmatrix}\right)$. Since π is a smooth representation, there is an m such that f_α is left-invariant by $K(m)$ where $K(m)$ is the principal congruence subgroup of level m. The proof follows using Proposition 1.7. □

Definition 4.1. For an irreducible admissible representation π of G, let $C(\pi)$ denote the least nonnegative integer k such that π admits a nonzero vector fixed under $\Gamma_0^1(k)$. This integer $C(\pi)$ is called the *conductor of π in the sense of new forms*. Fix an additive character ψ_F of F such that the largest fractional ideal of F on which ψ_F is trivial is $\mathcal{O}_F$. Let $C_e(\pi)$ be the integer c such that the epsilon factor $\epsilon(\pi, s, \psi_F)$ associated to π as in [10] is up to a constant q^{-cs}. This integer $C_e(\pi)$ is called the *conductor of π in the sense of epsilon factors*.

LEMMA 4.1. *Let π be any irreducible admissible representation of G. Let (π, V) be realized in its Kirillov model. Let $m \geq C(\pi)$. If $f \in V_m$, then*
(1) $f(xu) = f(x)$ *for all* $x \in \mathfrak{D}^*$ *and all* $u \in \mathfrak{O}^\times$,
(2) $\operatorname{supp}(f) \subset \mathfrak{P}^{1-d}$.

Proof. Since for any unit u, $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0^1(m)$, we get (1) using the formula in Lemma 3.1. Since $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0^1(m)$ for any $X \in \mathfrak{O}$, by Lemma 3.1, we have

$$\big(\Psi(XY) - 1\big) f(Y) = 0$$

for all $Y \in \mathfrak{D}^*$. So if $Y \notin \mathfrak{P}^{1-d}$, then it is possible to choose an $X \in \mathfrak{O}$ such that $\Psi(XY) \neq 1$, which implies that f vanishes on any such Y. This proves (2). □

LEMMA 4.2. *Let (π, V) be any irreducible admissible representation of G. Let (π, V) be realized in its Kirillov model. Let $c = C(\pi)$ be the conductor of π in the sense of new forms. Then we have the following.*
(1) *If $c = 0$ and if $0 \neq f \in V_0$, then $f(\varpi^{1-d}\mathfrak{O}^\times) \neq 0$.*
(2) *If $m \geq \max\{1, C(\pi)\}$ and if $f \in V_m$ is such that $f(\varpi^{1-d}\mathfrak{O}^\times) = 0$, that is, $\operatorname{supp}(f) \subset \mathfrak{P}^{2-d}$, then*

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi^{-1} \end{pmatrix} \cdot f \in V_{m-1}.$$

Proof. To prove (1), if possible let $0 \neq f \in V_0$ and $f(\varpi^{1-d}\mathfrak{O}^\times) = 0$. Observe that such an f is fixed by $N(\mathfrak{P}^{-1})$. Let H be the subgroup of G generated by $GL_2(\mathfrak{O})$ and $N(\mathfrak{P}^{-1})$. The matrix identity

$$\begin{pmatrix} 1 & \varpi^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varpi & 1 \end{pmatrix} = \begin{pmatrix} 0 & \varpi^{-1} \\ -\varpi & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \varpi \end{pmatrix} \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix}$$

gives that for all $n \in \mathbb{Z}$, the matrix $\left(\begin{smallmatrix} \varpi^{-n} & 0 \\ 0 & \varpi^{n} \end{smallmatrix}\right)$ is in H. Now choose any $A \in \mathfrak{P}^{2-d}$ such that $f(A) \neq 0$. Since $\left(\begin{smallmatrix} \varpi^{-n} & 0 \\ 0 & \varpi^{n} \end{smallmatrix}\right) f = f$, we get $f(\varpi^{-n} A \varpi^{-n}) \neq 0$. Choose n large enough to get a contradiction. This proves (1).

For the sake of brevity, let x denote the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$. Let $g = x^{-1} \cdot f$. So $\operatorname{supp}(g) \subset \mathfrak{P}^{1-d}$, which together with Lemma 3.1 implies that g is fixed by $N(\mathfrak{O})$. Since $f \in V_m$, we get that g is fixed by

$$x^{-1}\Gamma_0^1(m)x = \begin{bmatrix} \mathfrak{O}^\times & \mathfrak{P} \\ \mathfrak{P}^{m-1} & 1+\mathfrak{P}^m \end{bmatrix}.$$

So g is fixed by the subgroup of G generated by $x^{-1}\Gamma_0^1(m)x$ and $N(\mathfrak{O})$, which by Proposition 1.7 is the same as the subgroup of G generated by $\Gamma_0^1(m)$ and $\overline{N}(m-1)$, which by Proposition 1.8 is $\Gamma_0^1(m-1)$. This proves (2). □

LEMMA 4.3. *Let (π, V) be an irreducible admissible representation of G. For all $m \geq C(\pi)$, we have*

$$\dim_{\mathbb{C}}(V_m) - \dim_{\mathbb{C}}(V_{m-1}) \leq \dim_{\mathbb{C}}(V_{N,\Psi}).$$

Proof. As before, let x denote the matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)$. Clearly, $x \cdot V_{m-1}$ is fixed by $x\Gamma_0^1(m-1)x^{-1}$, which contains $\Gamma_0^1(m)$. Hence, $x \cdot V_{m-1} \subset V_m$. So to prove the lemma, it is enough to show that $\dim_{\mathbb{C}}(V_m/x \cdot V_{m-1}) \leq \dim_{\mathbb{C}}(V_{N,\Psi})$. Let $\dim_{\mathbb{C}}(V_{N,\Psi}) = r$. Let (π, V) be realized in its Kirillov model. Let $f_1, f_2, \ldots, f_{r+1}$ be $r+1$ vectors in V_m. By Lemma 4.1, we can choose $a_1, a_2, \ldots, a_{r+1} \in \mathbb{C}$ such that $f = \sum_{i=1}^{r+1} a_i f_i$ vanishes on $\varpi^{1-d}\mathbb{O}^\times$. By Lemma 4.2, we get that $f \in x \cdot V_{m-1}$. □

The following corollary of the proof of the previous lemma is improved later.

COROLLARY 4.1. *For any irreducible admissible representation of $\mathrm{GL}_2(\mathfrak{D})$, the space of new forms is a $\mathfrak{D}^*$-submodule of the twisted Jacquet module.*

4.1. New forms for principal series representations. In this section we study new forms and conductors for principal series representations.

LEMMA 4.4. *Let $K = \mathrm{GL}_2(\mathbb{O})$, and let $P(\mathbb{O})$ consist of all upper triangular matrices in K. For $i = 0, \ldots, m-1$, let $\gamma_i = \left(\begin{smallmatrix}1 & 0\\ \varpi^i & 1\end{smallmatrix}\right)$, and let $\gamma_m = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Then*

$$K = \amalg_{i=0}^m \Gamma_0^1(m)\gamma_i P(\mathbb{O}).$$

Proof. If $k = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in K$ and if $C \in \mathfrak{P}^i$ and $C \notin \mathfrak{P}^{i+1}$ for $1 \leq i \leq m$, then noting that $A \in \mathbb{O}^\times$ and $C\varpi^{-i} \in \mathbb{O}^\times$, we have

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \begin{pmatrix}AC^{-1}\varpi^i & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\begin{pmatrix}\varpi^{-i}C & \varpi^{-i}CA^{-1}B\\ 0 & D - CA^{-1}B\end{pmatrix},$$

which gives that $k \in \Gamma_0^1(m)\gamma_i P(\mathbb{O})$. If $C \in \mathbb{O}^\times$, multiplying k on the right by $\left(\begin{smallmatrix}C^{-1} & 0\\ 0 & C^{-1}\end{smallmatrix}\right)$, we may assume that $C = 1$. In this case, we have

$$\begin{pmatrix}A & B\\ 1 & D\end{pmatrix} = \gamma \cdot \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \cdot p,$$

where $\gamma = \left(\begin{smallmatrix}1-A\varpi^m & A(1+\varpi^m)-1\\ -\varpi^m & 1+\varpi^m\end{smallmatrix}\right)$ and $p = \left(\begin{smallmatrix}1 & D+(1+\varpi^m)(B-AD)\\ 0 & AD-B\end{smallmatrix}\right)$. □

LEMMA 4.5. *For $0 \leq i \leq m$, we have $\gamma_i^{-1}\Gamma_0^1(m)\gamma_i \cap P(\mathbb{O})$ is equal to*

$$\left\{\begin{pmatrix}A & B\\ 0 & D\end{pmatrix} \in K : \begin{array}{l}\text{(i)}\ \ A-1, B-(1-D)/\varpi^i \equiv 0 \left(\operatorname{mod} \mathfrak{P}^{m-i}\right)\\ \text{(ii)}\ \ D \equiv 1 \left(\operatorname{mod} \mathfrak{P}^i\right)\end{array}\right\}.$$

Proof. Let $x = \left(\begin{smallmatrix}A & B\\ 0 & D\end{smallmatrix}\right) \in P(\mathbb{O})$ be such that $\gamma_i x \gamma_i^{-1} \in \Gamma_0^1(m)$. Multiplying the matrices and using the defining congruences for $\Gamma_0^1(m)$ gives that $D + \varpi^i B - 1$ and $\varpi^i A - D\varpi^i - \varpi^i B\varpi^i$ are in $\mathfrak{P}^m$, which simplifies to what we want. □

THEOREM 4.1. *Let π_1 and π_2 be irreducible (necessarily finite-dimensional) representations of $\mathfrak{D}^*$. Let n_i be the smallest nonnegative integer such that π_i is trivial on $\mathfrak{D}^*(n_i)$, for $i = 1, 2$. Here $\mathfrak{D}^*(r)$ denotes the subgroup $1+\mathfrak{P}^r$ if $r \geq 1$ and $\mathbb{O}^\times$ if $r = 0$. Then we have the following.*

(1) *The smallest integer m such that $V(\pi_1, \pi_2)$ admits a $\Gamma_0^1(m)$-fixed vector is $m = n_1 + n_2$; that is, the conductor in the sense of new forms is given by*

$$C\big(V(\pi_1, \pi_2)\big) = n_1 + n_2.$$

(2) *Furthermore, $V(\pi_1, \pi_2)^{\Gamma_0^1(n_1+n_2)} \simeq \pi_1 \otimes \pi_2$ as $\mathfrak{D}^*$-modules.*

(3) *For $k \geq n_1 + n_2$, $V^{\Gamma_0^1(k)} \simeq (k - n_1 - n_2 + 1)(\pi_1 \otimes \pi_2)$ as $\mathfrak{D}^*$-modules.*

(4) *If $V(\pi_1, \pi_2)$ is irreducible, then the conductor in the sense of epsilon factors is given by*

$$C_e\big(V(\pi_1, \pi_2)\big) = n_1 + n_2 + 2(d-1).$$

Proof. Using Iwasawa decomposition (see Proposition 1.2), we have

$$V(\pi_1, \pi_2) = \mathrm{Ind}_{P(\mathbb{O})}^{K}(\pi_1 \otimes \pi_2).$$

By Mackey's theorem on restriction of an induced repesentation, the restriction of $V(\pi_1, \pi_2)$ to $\Gamma_0^1(m)$ is

$$\sum_{i=0}^{m} \mathrm{Ind}_{H_i}^{\Gamma_0^1(m)}(\pi_1 \otimes \pi_2),$$

where $H_i = P(\mathbb{O}) \cap \gamma_i^{-1} \Gamma_0^1(m) \gamma_i$ is considered to be a subgroup of $\Gamma_0^1(m)$ in a natural way. Therefore,

$$V(\pi_1, \pi_2)^{\Gamma_0^1(m)} \simeq \sum_{i=0}^{m} (\pi_1 \otimes \pi_2)^{H_i}.$$

Since $M \cap H_i$ is by the previous lemma $\mathfrak{D}^*(m-i) \times \mathfrak{D}^*(i)$, which is a normal subgroup of M, if $(\pi_1 \otimes \pi_2)^{H_i} \neq (0)$, then $m - i \geq n_1$ and $i \geq n_2$. This is possible if and only if $m \geq n_1 + n_2$, and if so, there are exactly $m - (n_1 + n_2) + 1$ many i's such that $(\pi_1 \otimes \pi_2)$ has an H_i-fixed vector.

Note that $\Delta\mathfrak{D}^*$ leaves $\Gamma_0^1(m)$ invariant, and hence $V(\pi_1, \pi_2)^{\Gamma_0^1(m)}$ has a $\mathfrak{D}^*$-module structure. It is easy to see that the above isomorphisms are $\mathfrak{D}^*$-equivariant. We have proved statements (1), (2), and (3).

To prove statement (4), we note that by [10] we have

$$\epsilon\big(V(\pi_1, \pi_2), s, \psi_F\big) = \epsilon(\pi_1, s, \psi_F) \cdot \epsilon(\pi_2, s, \psi_F).$$

It can be seen as in [14] that the exponent of q occurring in $\epsilon(\pi_i, s, \psi_F)$ is $n_i + d - 1$ for $i = 1, 2$. □

4.2. New forms for spherical representations in the Kirillov model. An irreducible admissible representation π of $G = \mathrm{GL}_2(\mathfrak{D})$ is said to be *spherical* if π admits a nonzero vector fixed by the maximal compact subgroup $K = \mathrm{GL}_2(\mathfrak{O})$. So in other words, spherical representations are exactly those whose conductor is 0 in the sense of new forms. It can be seen that an infinite-dimensional representation π is spherical if and only if π is an irreducible unramified principal series representation. See [23] for a proof of this statement.

Let π_1 and π_2 be 1-dimensional representations of $\mathfrak{D}^*$ that are trivial on $\mathfrak{O}^\times$. The representations π_i are therefore of the form $\pi_i(X) = \alpha_i(N_{\mathfrak{D}/F}(X))$ for unramified characters α_i of F^*. The representation $V(\pi_1, \pi_2)$ is called an unramified principal series representation of G. By Theorem 4.1, the principal series representation $V(\pi_1, \pi_2)$ has a $\mathrm{GL}_2(\mathfrak{O})$-fixed vector that is unique up to scalars. The aim of this section is to describe this vector, also called the *spherical vector*, in the Kirillov model.

THEOREM 4.2. *Let $\pi_i(X) = \alpha_i(N_{\mathfrak{D}/F}(X))$ for unramified characters α_i of F^* for $i = 1, 2$. The principal series representation $V(\pi_1, \pi_2)$ has a* $\mathrm{GL}_2(\mathfrak{O})$*-fixed vector that is unique up to scalars described in the Kirillov model as*

$$f(X) = |X|^{1/2} \sum_{\substack{i+j=\mathfrak{v}_{\mathfrak{D}}(X)+d-1 \\ i\geq 0,\, j\geq 0}} \alpha_1^i(\varpi_F)\alpha_2^j(\varpi_F)$$

if $\mathfrak{v}_{\mathfrak{D}}(x) \geq 1-d$, and $f(X) = 0$ otherwise.

Proof. We only sketch the argument that is exactly as in the field case as given, for instance, in [9].

Let $f \in \mathcal{K}(V(\pi_1, \pi_2))$ be the spherical vector in the Kirillov model. By Lemma 4.1, we know that f vanishes outside $\mathfrak{P}^{1-d}$, and the value of $f(X)$, for any $X \in \mathfrak{D}^*$, depends only on $|X|$. Hence, f is completely determined by the sequence of numbers $\{f_n\}_{n\geq 1-d}$, where $f_n = f(\varpi^n)$.

Let $y = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The charactersitic function χ of the double coset KyK is an element of the spherical Hecke algebra $C_c^\infty(K\backslash G/K)$. Hence, we get $\pi(\chi)f = cf$ for some constant c. Unraveling this while using the decomposition

$$KyK = \bigcup \begin{pmatrix} \varpi & * \\ 0 & 1 \end{pmatrix} K \cup \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} K,$$

where $*$ runs over representatives of $\mathfrak{O}/\mathfrak{P}$ gives a recurrence relation among the numbers f_n. Solving this recurrence relation gives the required formula. □

5. Supercuspidal representations. Let H be one of the two maximal compact modulo center subgroups defined in Proposition 1.4. We define a class of representations of H, called *very cuspidal*, that when induced to $\mathrm{GL}_2(\mathfrak{D})$ produce irreducible, supercuspidal representations. Both the definition and the proofs are exactly as in the field case. We would expect that all supercuspidal representations of $\mathrm{GL}_2(\mathfrak{D})$ can be obtained in this way, but we have not been successful in showing this. We begin with the definition of a very cuspidal representation. For this we must first define a filtration on H.

Define a filtration on $\mathcal{M}$ indexed by $\mathbb{Z}$ as

$$A_1(m) := \mathfrak{P}^m M_2(\mathbb{O}).$$

A decreasing filtration $H_1(m)$ on H_1 is now defined by

$$H_1(m) := 1 + A_1(m)$$

for all $m \geq 1$, and $H_1(0) := \mathrm{GL}_2(\mathbb{O}) = K_1$ (see Proposition 1.4). Similarly define two more filtrations on $\mathcal{M}$ as

$$A_2(m) := \mathfrak{P}^m \begin{bmatrix} \mathbb{O} & \mathbb{O} \\ \mathfrak{P} & \mathbb{O} \end{bmatrix}$$

and

$$B_2(m) := \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} A_2(m).$$

A decreasing filtration $H_2(m)$ is now defined by

$$H_2(m) := 1 + A_2(m)$$

for all $m \geq 1$; define $H_2(0) := K_2$ (see Proposition 1.4).

Definition 5.1. Let H be either H_1 or H_2. A finite-dimensional irreducible representation (σ, W) of H is called very cuspidal of level m if it is trivial on $H(m)$ and admits no nonzero vector fixed by the subgroup $N(\mathfrak{P}^{m-1}) \subset H/H(m)$.

With this definition, the following lemma is clear.

LEMMA 5.1. *Let (σ, W) be a very cuspidal representation of level m of H. Then σ restricted to $N(\mathbb{O})$ breaks up into eigencharacters of $\mathbb{O}$, all of which have conductor $\mathfrak{P}^m$, that is, trivial on $\mathfrak{P}^m$ and nontrivial on $\mathfrak{P}^{m-1}$. Furthermore, any such character occurs with the same multiplicity in σ. If we denote this common multiplicity as $r(\sigma)$, then the dimension of σ is given by*

$$\dim(\sigma) = r(\sigma) q^{d(m-1)} \big(q^d - 1\big).$$

LEMMA 5.2. *Let U denote the subgroup of G given by*

$$U := \left\{ \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} : u \in \mathbb{O}^\times \right\}.$$

Let (σ, W) be a very cuspidal representation of level m of H. Then

$$\dim\big(W^U\big) = r(\sigma).$$

Proof. Decompose σ as a sum of various characters of $N(\mathbb{O})$. Since the inner conjugation action of the image of U in H on $N(\mathbb{O})$ permutes the characters of $N(\mathbb{O})$ of conductor $\mathfrak{P}^m$ simply transitively, the lemma follows. □

PROPOSITION 5.1. *Let H denote either H_1 or H_2. Let (σ, W) and (σ', W') be two irreducible very cuspidal representations of level m of H. Let $(\pi, V) = \mathrm{ind}_H^G(\sigma)$, and let $(\pi', V') = \mathrm{ind}_H^G(\sigma')$. Then*

$$\dim\big(\mathrm{Hom}_G(\pi,\pi')\big) = \dim\big(\mathrm{Hom}_H(\sigma,\sigma')\big).$$

In particular, (π, V) is an irreducible admissible supercuspidal representation of G.

Proof. We use Kutzko's version of Mackey's theorem (see [15]), which describes the space of intertwining operators of two induced representations. For this we need a set of representatives of (H,H) double cosets in G, which by Proposition 1.3 can be taken to be $\left\{\begin{pmatrix}\varpi^a & 0\\ 0 & 1\end{pmatrix}\right\}_{a\ge 0}$. We have

$$\mathrm{Hom}_G(\pi,\pi') \subset \mathrm{Hom}_G\left(\mathrm{ind}_H^G(\sigma), \mathrm{Ind}_H^G(\sigma')\right) = \prod_{a\ge 0} \mathrm{Hom}_{H\cap g_a^{-1}\cdot H\cdot g_a}(\sigma, {}^{g_a}\sigma'),$$

where g_a denotes the matrix $\begin{pmatrix}\varpi^a & 0\\ 0 & 1\end{pmatrix}$. The following claim is proved exactly as in the field case.

Claim. If $\mathrm{Hom}_{H\cap g_a^{-1}\cdot H\cdot g_a}(\sigma, {}^{g_a}\sigma') \neq 0$, then $a = 0$.

The claim implies the proposition because then the only double coset that can support any nonzero intertwining operator is the identity double coset. □

The following easy lemma, whose proof is omitted, is used in the next proposition.

LEMMA 5.3. *With respect to the nondegenerate symmetric pairing on $\mathcal{M}$, given by $(X,Y) \mapsto \psi = \Psi(T_{\mathcal{M}/F}(XY))$, we have*

(1) $A_1(m)^\perp = A_1(-m+1-d)$,

(2) $A_2(m)^\perp = B_2(-m-d)$.

PROPOSITION 5.2. *For $i = 1$ or 2, let (σ, W) be a very cuspidal representation of H_i of level m, and let $(\pi, V) = \mathrm{ind}_{H_i}^G(\sigma)$. Then we have the following.*

(1) *The conductor of π in the sense of epsilon factors is*

$$C_e(\pi) = 2m + i - 1 + 2(d-1).$$

(2) *The representation π is minimal, that is, $C_e(\pi) \le C_e(\pi\otimes\chi)$ for all quasi characters χ of F^*.*

Proof. We start by computing the epsilon factor of π. We use the notation $f_1 \sim f_2$ for two functions of one complex variable s if there is a constant c such that $f_1(s) = cf_2(s)$ for all s. We refer the reader to [10] for notation and also a way of associating epsilon factors. It is easily checked that if π is an irreducible admissible supercuspidal representation of G, then so is its contragredient $\pi^\vee$, and for any such representation, the L-function associated to it as in [10] is the constant function 1. If we can find a function $\phi \in C_c^\infty(\mathcal{M})$ and a matrix coefficient f of π such that the associated zeta integral

$$Z(\phi, s, f) = \int_G \phi(x) f(x) \big|N_{\mathcal{M}/F}(x)\big|_F^s \, dx$$

is a constant, then by the functional equation, we would get that

$$\epsilon(\pi, s, \psi_F) \sim Z\big(\widehat{\phi}, -s, f^\vee\big)$$

where $\widehat{\phi}$ is the Fourier transform of ϕ with respect to the character ψ and $f^\vee$ is the function given by $f^\vee(g) = f(g^{-1})$ for all $g \in G$. We give the details in the case $i = 1$ and leave the other case to the reader since the argument is similar.

We choose ϕ to be the characteristic function of $H_1(m)$ thought of as a (compact open) subset of $\mathcal{M}$. Choose a vector $w \in W$ and a linear form w^* on W such that $\langle w, w^* \rangle = 1$. (Later on we choose this vector w more carefully.) Let f be the function on G that is 0 outside H_1 and is given by $f(h) = \langle \sigma(h^{-1})w, w^* \rangle$ on H_1; then f is a matrix coefficient of π. It is easily seen that for this choice of f and ϕ, the zeta integral $Z(\phi, s, f)$ is a constant by using the facts that reduced norm maps $H_1(m)$ into the group of units in F^* and that σ is trivial on $H_1(m)$.

Now it suffices to compute the integral $Z(\widehat{\phi}, -s, f^\vee)$. An easy computation using Lemma 5.3 gives that

$$\widehat{\phi}(x) = \begin{cases} 0, & x \notin A_1(-m+1-d), \\ \psi_F\big(T_{\mathcal{M}/F}(x)\big), & x \in A_1(-m+1-d). \end{cases}$$

So, abbreviating $T(x)$ for $T_{\mathcal{M}/F}(x)$, we have

$$Z\big(\widehat{\phi}, -s, f^\vee\big) = \int_{A_1(-m+1-d)\cap G} \psi_F\big(T(x)\big) f^\vee(x) \big|N_{\mathcal{M}/F}(x)\big|_F^{-s} \, dx.$$

Using the fact that $\mathrm{supp}(f) = H_1$, this integral can be split up as

$$\sum_{i=-m+1-d}^{\infty} \int_{\varpi^i H_1(0)} \psi_F\big(T(x)\big) f^\vee(x) \big|N_{\mathcal{M}/F}(x)\big|_F^{-s} \, dx.$$

By putting $x = \varpi^i y$, we get

$$\sum_{i=-m+1-d}^{\infty} q^{2is} \int_{H_1(0)} \psi_F\big(T\big(\varpi^i y\big)\big) f^\vee\big(\varpi^i y\big) \, dy.$$

Let $\mathcal{I}_i$ denote the integral occurring above.

Claim. For all $i \geq -m+2-d$, $\mathcal{I}_i = 0$.

Assuming the claim for the time being, we get that $\epsilon(\pi, s, \psi_F) \sim q^{2(-m+1-d)s}$ since, after all, the epsilon factor is some exponential and hence nonzero. From Definition 4.1 it follows that $C_e(\pi) = 2m+2(d-1)$. So it suffices to prove the claim.

An easy verification gives that

$$\mathcal{I}_i = \sum_{y \in H_1(0)/H_1(m)} \psi_F\left(T\left(\varpi^i y\right)\right) f^\vee\left(\varpi^i y\right).$$

This can be further split up as

$$\sum_{t \in H_1(0)/H_1(m-1)} \sum_{u \in H_1(m-1)/H_1(m)} \psi_F\left(T\left(\varpi^i t u\right)\right) f\left(u^{-1} t^{-1} \varpi^{-i}\right).$$

If $m \geq 2$, then choose w such that it is an eigenvector for $H_1(m-1)/H_1(m)$ with a nontrivial eigencharacter χ, and if $m = 1$, then we can take f restricted to $H_1(0)/H_1(1) = \mathrm{GL}_2(\mathbb{F}_{q^d})$ as a nontrivial matrix coefficient (of a cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_{q^d})$). We give further details for the case when $m \geq 2$ and leave the case of $m = 1$ to the reader. The previous expression for $\mathcal{I}_i$ can now be written by putting $u = 1+a$ where $a \in A_1(m-1)/A_1(m)$ as

$$\sum_{t \in H_1(0)/H_1(m-1)} f\left(t^{-1}\varpi^{-i}\right)\psi_F\left(T\left(\varpi^i t\right)\right) \sum_{a \in A_1(m-1)/A_1(m)} \psi_F\left(T\left(\varpi^i t a\right)\right)\chi'(a)$$

for an appropriate nontrivial character χ' on $A_1(m-1)/A_1(m)$. Now since $i \geq -m+2-d$, we get by Lemma 5.3 that $\psi_F(T(\varpi^i t a)) = 1$. This gives that $\mathcal{I}_i = 0$ since the inner summation is 0 as χ' is a nontrivial character on a finite abelian group $A_1(m-1)/A_1(m)$. This proves the claim. □

PROPOSITION 5.3. *Let H denote either H_1 or H_2. Let (σ, W) be an irreducible very cuspidal representation of level m of H, and let $(\pi, V) = \mathrm{ind}_H^G(\sigma)$. Let Ψ_m be the character on $N_0 = N(\mathcal{O})$ given by $\Psi_m\begin{pmatrix}1 & A\\ 0 & 1\end{pmatrix} = \Psi(\varpi^{-m+1-d}A)$. Then the natural map from W to V factors to give an isomorphism $W_{N_0,\Psi_m} \simeq V_{N,\Psi}$. In particular,* $\dim(V_{N,\Psi}) = r(\sigma)$.

Proof. It follows from Proposition 1.5 that $\left\{\begin{pmatrix}\varpi^a & 0\\ 0 & 1\end{pmatrix}\right\}_{a\in\mathbb{Z}}$ is a complete set of (N, H) double coset representatives. Using [15] and Frobenius reciprocity, we have

$$\begin{aligned}\mathrm{Hom}(\pi_{N,\Psi}, \mathbb{C}) &\simeq \mathrm{Hom}_N(\pi, \Psi)\\ &\simeq \mathrm{Hom}_G\left(\mathrm{ind}_H^G(\sigma), \mathrm{Ind}_N^G(\Psi)\right)\\ &\simeq \prod_{g \in N\backslash G/H} \mathrm{Hom}_{H\cap g^{-1}Ng}(\sigma, {}^g\Psi).\end{aligned}$$

Any $g = \begin{pmatrix} \varpi^a & 0 \\ 0 & 1 \end{pmatrix}$ normalizes N, and hence $H \cap g^{-1} N g = N(\mathbb{O})$. Furthermore, the character ${}^g\Psi$ on $N(\mathbb{O})$ is given by

$$ {}^g\Psi\left(\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}\right) = \Psi\left(g \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} g^{-1}\right) = \Psi(\varpi^a A). $$

This gives that the conductor of ${}^g\Psi$ is equal to $-a+1-d$. From Lemma 5.1 we know that σ restricted to $N(\mathbb{O})$ breaks up into eigencharacters, all of which have conductor $\mathfrak{P}^m$. This gives that the only double coset that can support a nonzero intertwining operator is when $a = -m+1-d$, and for this value of a we have ${}^g\Psi = \Psi_m$. We therefore get that

$$ \mathrm{Hom}(\pi_{N,\Psi}, \mathbb{C}) \simeq \mathrm{Hom}_{N(\mathbb{O})}(\sigma, \Psi_m) \simeq \mathrm{Hom}(\sigma_{N_0,\Psi_m}, \mathbb{C}), $$

or by dualizing we get

$$ \pi_{N,\Psi} \simeq \sigma_{N_0,\Psi_m}. $$

By Lemma 5.1 we also get $\dim(\pi_{N,\Psi}) = r(\sigma)$. □

PROPOSITION 5.4. *For $i = 1$ or 2, let (σ, W) be an irreducible very cuspidal representation of level m of H_i, and let $(\pi, V) = \mathrm{ind}_{H_i}^G(\sigma)$. Then*

(1) $V^{\Gamma_0^1(2m+i-2)} = (0)$,
(2) $\dim(V^{\Gamma_0^1(2m+i-1)}) = \dim(V_{N,\Psi})$,
(3) $C(\pi) = 2m+i-1$.

Proof. We consider only $H = H_1$, the other case being similar. Since $G = HAH$ where $A = \left\{\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}, n \geq 0\right\}$, it follows from Mackey theory that the restriction of $\mathrm{ind}_H^G(\sigma)$ to H is a sum of representations

$$ W_n = \mathrm{ind}_{\Gamma_0(n)}^H(\sigma). $$

From the definition of very cuspidality, it is easy to see that W_n is a representation of H of level $m+n$, and that the possible conductors of characters of $N(\mathbb{O})$ appearing in W_n lie between $\max\{0, m-n\}$ and $m+n$. Since $\Gamma(2m-1) \subset \Gamma_0^1(2m-1)$, this proves (1). Now observe that conjugation by $\begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix}$ takes $\Gamma_0^1(2m)$ into the subgroup U of H, whose image in $H/H(m)$ consists of diagonal entries with 1 at the right-hand bottom corner. Lemma 5.2 therefore implies that $\dim(V^{\Gamma_0^1(2m+i-1)}) \geq \dim(V_{N,\Psi})$, and hence (2) follows by Lemma 4.3. □

We summarise our analysis of new forms and conductors in the following two theorems. *These theorems have been proved only for those representations that are either principal series or are supercuspidal representations which are induced from a very cuspidal representation of a maximal compact mod center subgroup. We expect these theorems to be true for all irreducible infinite-dimensional representations.*

THEOREM 5.1. *Let (π, V) be an irreducible admissible infinite-dimensional representation of G. Then the space of new forms $V_{C(\pi)}$ is isomorphic as $\mathfrak{D}^*$-modules to the twisted Jacquet module $V_{N,\Psi}$.*

THEOREM 5.2. *Let π be an irreducible admissible representation of $\mathrm{GL}_2(\mathfrak{D})$. Then the conductor in the sense of new forms is related to the conductor in the sense of epsilon factors by the formula*

$$C_e(\pi) - C(\pi) = 2(d-1),$$

where d is the index of the division algebra $\mathfrak{D}$ over F.

Theorem 5.1 can be strengthened as follows.

THEOREM 5.3. *Let (π, V) be an irreducible admissible representation that is either a principal series representation or a supercuspidal representation that is compactly induced from a very cuspidal representation of a maximal open compact mod center subgroup. Let x denote the element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$. Then for all $m \geq 0$,*

$$V_{c+m} \simeq V_c \oplus x \cdot V_c \oplus \cdots \oplus x^m \cdot V_c,$$

$$\dim_{\mathbb{C}}(V_{c+m}) = (m+1) \dim_{\mathbb{C}}(V_{N,\Psi}).$$

Proof. The proof is by induction on m. The case $m = 0$ follows from Theorem 5.1. We prove that $V_{c+m} = V_{c+m-1} \oplus x^m \cdot V_c$, which completes the proof of the theorem. Since $x^m \Gamma_0^1(c) x^{-m}$ contains $\Gamma_0^1(c+m)$, we have that $x^m \cdot V_c$ is contained in V_{c+m}. So by Lemma 4.3 and Theorem 5.1, the theorem follows if we show that any $v \in V_c$ for which $x^m \cdot v \in V_{c+m-1}$ is necessarily trivial. But this follows from Proposition 1.8 since if G_v is the stabilizer in G of such a vector v, then G_v contains $\Gamma_0^1(c)$ and also $x^{-m} \Gamma_0^1(c+m-1) x^m$, which in turn contains $\overline{N}(c-1)$. □

Our final result in this section identifies new forms of supercuspidal representations explicitly in the Kirillov model.

PROPOSITION 5.5. *For $i = 1$ or 2, let (σ, W) be an irreducible very cuspidal representation of level m of H_i, and let $(\pi, V) = \mathrm{ind}_{H_i}^G(\sigma)$. Let $c = C(\pi) = 2m - 1 + i$ be the conductor of π in the sense of new forms. Then*

$$V_c = \left\{ f \in \mathcal{K}(\pi) : \begin{array}{ll} \text{(i)} & f(Xu) = f(X), \ \forall u \in \mathbb{O}^\times, \forall X \in \mathfrak{D}^* \\ \text{(ii)} & \mathrm{supp}(f) \subset \varpi^{1-d} \mathbb{O}^\times \end{array} \right\},$$

where we recall that $V_c = V^{\Gamma_0^1(c)}$.

Proof. We give the details for the case $i = 1$ and leave the other case, which is very similar, to the reader. For $w \in W^U$, let $g_w \in V$ be given by

$$g_w(x) = \begin{cases} 0, & x \notin H_1, \\ \sigma(x) w, & x \in H_1. \end{cases}$$

Also, let

$$f_w := \begin{pmatrix} \varpi^{-m} & 0 \\ 0 & 1 \end{pmatrix} g_w.$$

So f_w is a new form for π, and by Lemma 5.2 and Proposition 5.4, these are all the new forms of π. By Lemma 4.1 and Theorem 5.3, it is enough to show that the vector ϕ_w in the Kirillov space $\mathcal{K}(\pi)$ corresponding to f_w vanishes on $\mathfrak{P}^{2-d}$. Let $X \in \mathfrak{P}^{2-d}$, so we need to show that $\phi_w(X) = 0$.

From Section 3 we need to show that $\pi\left(\begin{smallmatrix} X & 0 \\ 0 & 1 \end{smallmatrix}\right) f_w \in V(N, \Psi)$. Hence, it is enough to show that there exists a $k \in \mathbb{Z}$ such that

$$\int_{\mathfrak{P}^k} \overline{\Psi(Y)} \left(\begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} f_w \right)(x)\, dY = 0, \quad \forall x \in G.$$

We may write $X = u\varpi^r$ where $r \geq 2-d$ and $u \in \mathbb{O}^\times$. Using the definitions of f_w and g_w, it is enough to show that there exists a $k \in \mathbb{Z}$ such that

$$\int_{\mathfrak{P}^k} \overline{\Psi(Y)} g_w \left(x_u \begin{pmatrix} \varpi^{r-m} & Y \\ 0 & 1 \end{pmatrix} \right) dY = 0, \quad \forall x \in G,$$

where $x_u = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} x \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is clear that as x varies over G, so does x_u. Using Proposition 1.5, we may write $G = H_1 \cdot P_1$, where $P_1 = \left\{\left(\begin{smallmatrix} \varpi^a & Z \\ 0 & 1 \end{smallmatrix}\right) : a \in \mathbb{Z},\ Z \in \mathfrak{D}\right\}$. Now by the definition of g_w and this decomposition for G, it is enough to show that there exists a $k \in \mathbb{Z}$ such that

$$\int_{\mathfrak{P}^k} \overline{\Psi(Y)}\, g_w \left(\begin{pmatrix} \varpi^a & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi^{r-m} & Y \\ 0 & 1 \end{pmatrix} \right) dY = 0, \quad \forall a \in \mathbb{Z},\ \forall Z \in \mathfrak{D}.$$

Since $\operatorname{supp}(g_w) = H_1$ and since $\left(\begin{smallmatrix} \varpi^t & T \\ 0 & 1 \end{smallmatrix}\right) \in H_1$ if and only if $t = 0$ and $T \in \mathbb{O}$, it is enough to show that there exists a $k \in \mathbb{Z}$ such that

$$\int_{\mathfrak{P}^k} \overline{\Psi(Y)}\, g_w \begin{pmatrix} 1 & \varpi^{m-r} Y + Z \\ 0 & 1 \end{pmatrix} dY = 0, \quad \forall Z \in \mathfrak{P}^{m-r+k}.$$

Making the substitution $T = \varpi^{m-r} Y + Z$, we need to show that there exists $k \in \mathbb{Z}$ such that

$$\int_{\mathfrak{P}^{m-r+k}} \overline{\Psi(\varpi^{r-m} T)}\, g_w \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} dT = 0.$$

Now choose $k = r-1$ and since $r \geq 2-d$ and the conductor of Ψ is $\mathfrak{P}^{1-d}$, the above integral boils down to

$$\int_{\mathfrak{P}^{m-1}} g_w \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} dT = \int_{\mathfrak{P}^{m-1}} \sigma \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} w\, dT = 0,$$

by definition of very cuspidality of σ. □

Remark 5.1. In this article we deal with a family of supercuspidal representations of G which are obtained by compactly inducing very cuspidal representations of maximal open compact subgroups. We follow [16] for this. We may also follow [6] and get another notion of very cuspidal representation of these subgroups, which also compactly induce to give irreducible admissible supercuspidal representations of G. This approach is taken in [23], to which the reader may refer for details. This other approach also lends itself to all the computations made here, namely, determining the twisted Jacquet module, computing the epsilon factors (really the exponent occuring in these factors), and determining the space of new forms and also the conductor in the sense of new forms.

Remark 5.2. Following Howe's construction of supercuspidal representations of $\mathrm{GL}_n(F)$ (see [11]), we may in the tame case (i.e., when p does not divide $2d$ where p is the characteristic of the residue field of F) attach supercuspidal representations of G to "admissible characters" of degree $2d$ field extensions of F. (This is expected; as in the tame case, via the local Langlands correspondence, an irreducible representation of the Weil group W_F of F is monomial. See [25].) See especially [18] for a lucid discussion of this theme.The second author is filling in the details of this approach and the results will appear elsewhere.

Remark 5.3. There is an elaborate theory by C. J. Bushnell and P. C. Kutzko in [5] that describes the theory of types for $\mathrm{GL}_n(F)$. P. Broussous has been adapting their formalism in the context of forms of $\mathrm{GL}(n)$. The reader is referred to his papers, particularly [2], which applies to our context and gives another approach to handle supercuspidal representations of G.

6. Shalika model. There is a notion called the Shalika model that is closely related to the Kirillov model. Note that in general we do not have a multiplicity-1 theorem for the Kirillov model, but if we add a further condition on the model we are interested in, then we do get a multiplicity-1 theorem, but the price we pay is that only a small number of representations admit such a model. In this section we consider the notion of Shalika models and prove that an irreducible representation π admits at most one Shalika model and if it does admit a Shalika model, then π is self-contragredient. The reader is referred to the article of Jacquet and Rallis [13] where analogous notions are considered for $\mathrm{GL}(2n, F)$.

Definition 6.1. A linear functional $\ell : V \to \mathbb{C}$ for an irreducible admissible representation (π, V) of G is called a *Shalika functional* if

$$\ell\left(\pi\begin{pmatrix} A & X \\ 0 & A \end{pmatrix} v\right) = \Psi\big(A^{-1}X\big)\ell(v)$$

for all $A \in \mathcal{D}^*, X \in \mathcal{D}$, and $v \in V$. The space of all Shalika functionals for π is denoted by $\mathcal{S}_\pi$.

Before we come to the statement of the main result of this section, we recall the following theorem of the first author (see [20] and [21]).

THEOREM 6.1. *Let π be an irreducible admissible representation of $G = \mathrm{GL}_2(\mathcal{D})$, and let M be the subgroup of diagonal matrices. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_M(\pi, \mathbb{C}) \leq 1.$$

Furthermore, if $\dim_{\mathbb{C}} \mathrm{Hom}_M(\pi, \mathbb{C}) = 1$, *then* $\pi \simeq \pi^\vee$.

The following is the main theorem of this section.

THEOREM 6.2. *If (π, V) is an irreducible admissible representation of* $\mathrm{GL}_2(\mathcal{D})$, *then the dimension of the space of Shalika functionals is at most* 1, *that is,*

$$\dim_{\mathbb{C}}(\mathcal{S}_\pi) \leq 1.$$

Furthermore, if $\dim_{\mathbb{C}}(\mathcal{S}_\pi) = 1$, *then* $\pi \simeq \pi^\vee$.

Proof. By Theorem 2.1 we know explicitly the twisted Jacquet module of any parabolically induced representation, and the above theorem falls out as an easy corollary for any subquotient of such a representation.

We can therefore assume in the rest of the proof that π is a supercuspidal representation for which we construct an injective linear map from $\mathcal{S}_\pi$ to $\mathrm{Hom}_M(\pi, \mathbb{C})$, which by the earlier theorem, completes the proof of this theorem.

To construct an injective map from $\mathcal{S}_\pi$ to $\mathrm{Hom}_M(\pi, \mathbb{C})$, we note that a Shalika functional ℓ is in particular a degenerate Whittaker functional. Therefore, we have an S-invariant linear map $\pi_{N,\psi} \to \mathbb{C}$, which when composed with the natural map from π to $\pi_{N,\Psi}$ gives rise to ℓ. We therefore have the following sequence of homomorphisms:

$$\pi \longrightarrow C_c^\infty\big(\mathcal{D}^*, \pi_{N,\psi}\big) \longrightarrow C_c^\infty(\mathcal{D}^*) \longrightarrow \mathbb{C},$$

where the map $C_c^\infty(\mathcal{D}^*) \to \mathbb{C}$, is given by integrating over $\mathcal{D}^*$. The first arrow above is an isomorphism, and the second is a surjection. The composite of all the three maps gives rise to a nonzero M-invariant linear map on π, completing the proof of the theorem. □

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