# RELATING INVARIANT LINEAR FORM AND LOCAL EPSILON FACTORS VIA GLOBAL METHODS

DIPENDRA PRASAD

Appendix by HIROSHI SAITO

**Abstract**

*We use the recent proof of Jacquet's conjecture due to Harris and Kudla [HK] and the Burger-Sarnak principle (see [BS]) to give a proof of the relationship between the existence of trilinear forms on representations of* $\mathrm{GL}_2(k_u)$ *for a non-Archimedean local field* $k_u$ *and local epsilon factors which was earlier proved only in the odd residue characteristic by this author in [P1, Theorem 1.4]. The method used is very flexible and gives a global proof of a theorem of Saito and Tunnell about characters of* $\mathrm{GL}_2$ *using a theorem of Waldspurger [W, Theorem 2] about period integrals for* $\mathrm{GL}_2$ *and also an extension of the theorem of Saito and Tunnell by this author in [P3, Theorem 1.2] which was earlier proved only in odd residue characteristic. In the appendix to this article, H. Saito gives a local proof of Lemma 4 which plays an important role in the article.*

## 1. Triple products

Let $\pi_1, \pi_2$, and $\pi_3$ be three irreducible admissible infinite-dimensional representations of $\mathrm{GL}_2(k_u)$ for a non-Archimedean local field $k_u$ with the product of their central characters trivial. Let $D_u$ denote the unique quaternion division algebra over $k_u$. For an irreducible admissible discrete series representation $\pi$ of $\mathrm{GL}_2(k_u)$, let $\pi'$ denote the representation of $D_u^*$ associated to $\pi$ by the Jacquet-Langlands correspondence, and let $\pi' = 0$ if $\pi$ is not a discrete series representation.

The author in [P1] studied the space of trilinear forms $\ell : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ which are $\mathrm{GL}_2(k_u)$-invariant. Let $m(\pi_1 \otimes \pi_2 \otimes \pi_3)$ denote the dimension of the space of such trilinear forms, and let $m(\pi_1' \otimes \pi_2' \otimes \pi_3')$ denote the dimension of the space of $D_u^*$-invariant linear forms on $\pi_1' \otimes \pi_2' \otimes \pi_3'$ (so $m(\pi_1' \otimes \pi_2' \otimes \pi_3')$ is nonzero only if all the $\pi_i$ are discrete series representations). We let $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = \epsilon(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)$ denote the triple product epsilon factor defined by the Langlands-Shahidi method (cf. [Sh, Section 4]); under the condition that the product of the central characters is trivial, $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = \pm 1$. Usually, the epsilon factor

DUKE MATHEMATICAL JOURNAL
Vol. 138, No. 2, © 2007
Received 5 February 2006. Revision received 29 August 2006.
2000 *Mathematics Subject Classification*. Primary 22E50; Secondary 11F70.

depends on an auxiliary additive character of the field, but in our case, it is independent of it.

The main results proved in [P1] are

(1) *multiplicity one theorem:* $m(\pi_1 \otimes \pi_2 \otimes \pi_3) \leq 1$;

(2) *dichotomy principle:* $m(\pi_1 \otimes \pi_2 \otimes \pi_3) + m(\pi_1' \otimes \pi_2' \otimes \pi_3') = 1$;

(3) *theorem about epsilon factors:* $m(\pi_1 \otimes \pi_2 \otimes \pi_3) = 1$ if and only if $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = 1$.

A few words about the proofs. The multiplicity one theorem was proved by the method of *Gelfand pairs*, as developed by Gelfand and Kazhdan, and was thus based on general principles available to prove such theorems.

The dichotomy principle was eventually related to the character identity

$$\Theta_\pi(x) = -\Theta_{\pi'}(x)$$

at regular elliptic elements $x$ for discrete series representations $\pi$ and $\pi'$ of $\mathrm{GL}_2(k_u)$ and $D_u^*$, respectively, associated to each other by the Jacquet-Langlands correspondence. This, combined with the theorem on finite groups, according to which

$$m(\pi_1' \otimes \pi_2' \otimes \pi_3') = \frac{1}{\mathrm{vol}(D_u^*/k_u^*)} \int_{D_u^*/k_u^*} \Theta_{\pi_1'}(g)\Theta_{\pi_2'}(g)\Theta_{\pi_3'}(g)\, dg,$$

and a suitable variant for $\mathrm{GL}_2(k_u)(!)$, proves the dichotomy principle for supercuspidal representations, others being much more straightforward by the orbit method of Mackey.

These two theorems, that is, the multiplicity one theorem and the dichotomy principle, had reasonably satisfactory proofs. However, the theorem about epsilon factors was proved case by case, reducing it to simpler epsilon factors studied by Tunnell [T], and this reduction was possible only for representations of $\mathrm{GL}_2(k_u)$ arising from characters of quadratic extensions of $k_u$ and was thus incomplete in even residue characteristic. Also, the proof of the theorem on epsilon factors given in [P1, Theorem 1.4] left much to be desired, being a brute-force calculation that essentially amounted to calculating the epsilon factor of the triple product on the one hand, relating it to $m(\pi_1 \otimes \pi_2 \otimes \pi_3)$ through a totally independent calculation, and then observing that the results are the same. This was done via explicit knowledge of the character of representations of $\mathrm{GL}_2(k_u)$, which has been known for a long time through the work of Sally and Shalika, the epsilon factor (associated to Galois representations) being calculated either directly or through Tunnell's work, which also was a brute-force calculation with characters. (Later, there was the elegant article of Saito [S] which proved Tunnell's theorem in all residue characteristics.)

The aim of this article is to offer a global method for the result on epsilon factors using the fact that the global analogue of the results on trilinear form, which is the period integral, has recently been proved by Harris and Kudla [HK]. (This result was conjectured by H. Jacquet.) More important, the present proof, unlike the earlier one, offers a conceptual reason why the theorem on epsilon factors holds good and suggests that the natural proof of the general conjectures of Gross and Prasad in [GP1] about local branching laws from $\mathrm{SO}(n)$ to $\mathrm{SO}(n-1)$ in terms of epsilon factors should be through global means. This approach is especially promising since the global conjecture in [GP1] about nonvanishing of the period integral in terms of an $L$-value at $1/2$, it seems, can be proved by refining the work [GJR] (see Section 6 for some more details on these general conjectures).

We remark that since the global theorems used in this work have been considered only for number fields so far, we will be able to deduce local theorems only in characteristic 0; hopefully, this situation will be remedied in the near future.

We now recall the theorem of Harris and Kudla.

THEOREM 1 (see Harris and Kudla [HK])
*Let $k$ be a number field, and let $\Pi_1$, $\Pi_2$, and $\Pi_3$ be three cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_k)$ with the product of their central characters trivial. For a quaternion algebra $D$ over $k$, let $\Pi_i^D$ be the automorphic representations of $(D\otimes_k \mathbb{A}_k)^*$ associated to $\Pi_i$ by the global Jacquet-Langlands correspondence, if it exists. Then the central critical $L$-value $L(1/2, \Pi_1 \otimes \Pi_2 \otimes \Pi_3)$ is nonzero if and only if, for some $D$, $\Pi_i^D$ exist as automorphic representations and there are $f_i^D \in \Pi_i^D$ such that*

$$\int_{D^*\mathbb{A}_k^*\backslash(D\otimes_k\mathbb{A}_k)^*} f_1^D(g) f_2^D(g) f_3^D(g)\, d^\times g \neq 0.$$

The proof of Theorem 1 does not use the theorem on epsilon factors, so it is legitimate to use it to prove the following theorem.

THEOREM 2
*Let $\pi_1$, $\pi_2$, and $\pi_3$ be three irreducible admissible infinite-dimensional representations of $\mathrm{GL}_2(k_u)$ for a non-Archimedean local field $k_u$ (of characteristic 0) with the product of their central characters trivial. Then there exists a nonzero $\ell : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ which is $\mathrm{GL}_2(k_u)$-invariant if and only if $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = 1$.*

*Proof*
If one of the representations $\pi_i$ is either a principal series or a twist of the Steinberg representation, then the result can be proved by simple calculations as in [P1]. (Essentially because in all these cases, except when one is dealing with the triple product of the Steinberg, there is an invariant linear form as follows by simple orbit methods; also, the epsilon factor is easily calculated to be 1 in all these cases.) We therefore assume in the rest of this proof that all the representations $\pi_i$ are supercuspidal representations of $\mathrm{GL}_2(k_u)$ with the product of their central characters trivial. (Of course, global methods are inadequate to deal with principal series representations, as they may not appear as a local component of a global representation.)

We first prove that if there is a nonzero $\ell : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ which is $\mathrm{GL}_2(k_u)$-invariant, then $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = 1$. Then we use the dichotomy principle to say that if the space of $\mathrm{GL}_2(k_u)$-invariant forms on $\pi_1 \otimes \pi_2 \otimes \pi_3$ is zero, then $\pi_i$ are all discrete series representations, and for the corresponding representations $\pi_i'$ of $D_u^*$, where $D_u^*$ is the unique quaternion division algebra over $k_u$, there is a $D_u^*$-invariant linear form on $\pi_1' \otimes \pi_2' \otimes \pi_3'$. Now, by the same method employed to prove the first case (when there is a $\mathrm{GL}_2(k_u)$-invariant linear form), we prove that the local epsilon factor $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3)$ in this case is $-1$. It enables us to complete the proof of the theorem.

We begin with the case when there is a nonzero $\ell : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ which is $\mathrm{GL}_2(k_u)$-invariant and prove that $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = 1$. Fix a totally real number field $k$ and a place $u$ of $k$ so that the completion of $k$ at $u$ is the local field $k_u$ with which we started. Let $D$ be a quaternion division algebra over $k$ which is unramified at all the finite places, and for simplicity, we assume that $D$ is ramified at all the infinite places. This is done by choosing $k$ appropriately so that it has even degree over $\mathbb{Q}$.

We let $\Pi_1$ and $\Pi_2$ be automorphic representations of $D(\mathbb{A}_k)^*$ with local components $\pi_1$ and $\pi_2$, respectively, at $u$, unramified at all the other finite places outside $u$. It is well known that local supercuspidal representations can be obtained as the local component of an automorphic representation, which is unramified at all the other finite places, and have some weights at infinity.

We choose $\Pi_3$, which has $\pi_3$ as its local component at $u$ (and some infinity type). The representation $\Pi_3$ is constructed so that the period integral

$$\int_{D^* \mathbb{A}_k^* \backslash D^*(\mathbb{A}_k)} f_1(g) f_3(g) f_3(g)\, d^\times g$$

is nonzero for some choice of functions $f_i \in \Pi_i \subset L^2(D^* \backslash D^*(\mathbb{A}_k))$. By the general lemma, Lemma 1, which is part of Burger-Sarnak philosophy, such choices can be made.

Once we have automorphic representations $\Pi_i$ with nonvanishing period integral, the theorem of Harris and Kudla implies that

$$L\Big(\frac{1}{2}, \Pi_1 \otimes \Pi_2 \otimes \Pi_3\Big) \neq 0.$$

This implies, in particular, that the global sign $\epsilon(1/2, \Pi_1 \otimes \Pi_2 \otimes \Pi_3)$ in the functional equation for $L(s, \Pi_1 \otimes \Pi_2 \otimes \Pi_3)$ is 1. However, the global sign in the functional equation is nothing but the product of the local epsilon factors. From the information that $\Pi_1$ is a principal series representation at all the finite places of $k$ except $u$, it is easy to see that the epsilon factor at all the finite places except $u$ is 1; this follows from the general fact that $\epsilon(\sigma)\epsilon(\sigma^\vee) = \det(\sigma)(-1)$.

The triple product epsilon factor at all the infinite places is $-1$ by invoking the corresponding theorem at infinity (the local representations at infinity are given to have invariant linear forms); these calculations at infinity are simple consequences of results about the decomposition of the tensor product of finite-dimensional representations of $D^*(\mathbb{R})$, the so-called Clebsch-Gordon theorem, and are discussed in [P1, Section 9].

It is clear, then, from the factorisation

$$\epsilon\Big(\frac{1}{2}, \Pi_1 \otimes \Pi_2 \otimes \Pi_3\Big) = \prod_{\mu} \epsilon\Big(\frac{1}{2}, \Pi_{1,\mu} \otimes \Pi_{2,\mu} \otimes \Pi_{3,\mu}\Big) = 1,$$

and recalling that there is an even number of places at infinity, we conclude that $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = 1$, thus proving that if the space of $\mathrm{GL}_2(k_u)$-invariant linear forms on $\pi_1 \otimes \pi_2 \otimes \pi_3$ is nonzero, the triple product epsilon factor $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3)$ is 1.

Assume now that there is no nonzero $\ell : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ which is $\mathrm{GL}_2(k_u)$-invariant. By the dichotomy theorem, the representations $\pi_i$ are discrete series representations of $\mathrm{GL}_2(k_u)$, and there is a nonzero $\ell' : \pi_1' \otimes \pi_2' \otimes \pi_3' \to \mathbb{C}$ which is $D_u^*$-invariant. By exactly similar analysis, we prove that $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = -1$. For this, choose a quaternion division algebra over $k$ which is ramified exactly at $u$ and at all the infinite places of $k$ which are now assumed by choosing the number field $k$ appropriately to be odd in number. Once again, the sign in the functional equation of the triple product $L$-function is 1, and since the theorem at infinity gives the sign $-1$ at each of the infinite places, we get in this case $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = -1$, completing the proof of the theorem. □

Let us emphasize that in our proof, the local epsilon factor at a finite place is matched to one at infinity by global means, reducing us to a much simpler problem.

*Remark 1*
The epsilon factors used in [P1] were those arising from Galois representations, whereas the epsilon factors used here are those defined by the Langlands-Shahidi method. It is a theorem of Ramakrishnan (cf. [R, Theorem 4.4.1]) that these two epsilon factors are the same.

The following lemma is essentially due to Burger and Sarnak [BS], but not quite. A result of this kind only for the infinite prime is [HL, Proposition 3.1]. Adding finite primes causes no extra difficulty. However, for the sake of completeness, we give a self-contained proof of a result which is adequate for our purposes and which is an elementary consequence of the weak approximation theorem (which we assume holds good for our group $G$, which is certainly the case in all our applications).

LEMMA 1

*Let $k$ be a number field, let $S$ be a finite set of finite places of $k$, let $G$ be a reductive algebraic group defined over $k$, and let $H$ be a reductive subgroup of $G$. Suppose that $Z$ is a central subgroup of $H$ which remains central in $G$ with the property that $Z\backslash H$ has no nontrivial $k$-rational characters. Let $G_S = \prod_{v\in S} G(k_v)$; similarly, let $H_S = \prod_{v\in S} H(k_v)$. Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation of $G(\mathbb{A}_k)$. Suppose that $\mu_v$ are supercuspidal representations of $H(k_v)$, $v \in S$, which are induced from representations $\nu_v$ of subgroups $\mathscr{K}_v$ which are certain open subgroups of $H_v$ compact modulo $Z_v$. Assume that $\mu_v$ appears as a quotient of $\pi_v$ restricted to $H(k_v)$ for all $v \in S$. Then there are an automorphic representation $\mu = \bigotimes_v \mu_v$ of $H(\mathbb{A}_k)$ with $\mu_S = \bigotimes_{v\in S} \mu_v$ and functions $f_1 \in \pi$, $f_2 \in \mu$ such that*

$$\int_{H(k)Z(\mathbb{A}_k)\backslash H(\mathbb{A}_k)} f_1 \bar{f}_2 \, dh \neq 0.$$

*Proof*

By the assumption that $\mu_v$ appears as a quotient of $\pi_v$ restricted to $H(k_v)$ for all $v \in S$, the representation $\nu_v$ of $\mathscr{K}_v$ is a subrepresentation of $\pi_v$ restricted to $\mathscr{K}_v$ for all $v \in S$. This means that there is a function $f$ on $G(k)\backslash G(\mathbb{A}_k)$ whose ($\mathscr{K}_S = \prod_{v\in S} \mathscr{K}_v$)-translates generate a space of functions which is isomorphic to $\bigotimes \nu_v$ as $\mathscr{K}_S$-modules. We prove that the restriction of *one* such function to $H(k)\backslash H(\mathbb{A}_k)$ is not identically zero. Observe that $G(\mathbb{A}^S) = \left\{x \in G(\mathbb{A}), x = \prod_v x_v \,\middle|\, x_v = 1, \forall v \in S\right\}$ operates on such functions (by right translation), and if all the $G(\mathbb{A}^S)$-translates of a function $f$ were zero at the identity element of $G(\mathbb{A})$, the function $f$ would be identically zero by the weak approximation theorem, according to which $G(k)$ is dense in $G_S$. Thus we have a function $f$ on $G(k)\backslash G(\mathbb{A}_k)$ whose restriction to $H(k)\backslash H(\mathbb{A}_k)$, say, $\tilde{f}$, is not zero. The $\mathscr{K}_S$-translates of $\tilde{f}$ generate a space of functions now on $H(k)\backslash H(\mathbb{A}_k)$ which is isomorphic to $\bigotimes \nu_v$ as $\mathscr{K}_S$-modules. Since

$$\mu_S = \operatorname{ind}_{\mathscr{K}_S}^{H_S} \nu_S,$$

the $H_S$-translates of $\tilde{f}$ generate a space of functions on $H(k)\backslash H(\mathbb{A}_k)$ which are isomorphic to $\mu_S$ as $H_S$-modules. We are now done, by Lemma 2. □

LEMMA 2

*Suppose that $H$ is a reductive algebraic group over a number field $k$, and suppose that $Z$ is a central subgroup in $H$ with the property that $Z\backslash H$ has no nontrivial $k$-rational characters. Suppose that $S$ is a finite set of finite places and that $H_S = \prod_{v\in S} H(k_v)$. Suppose that $\tilde{f}$ is a bounded continuous function on $H(k)\backslash H(\mathbb{A}_k)$ on which $Z(\mathbb{A}_k)$ operates via a unitary central character $\chi$ and whose $H_S$-translates generate an irreducible $H_S$-submodule, say, $\mu_S$, of a space of functions on $H(k)\backslash H(\mathbb{A}_k)$. Assume* *that $\mu_S$ is a supercuspidal representation of $H_S$. Then there exists a cusp form $g$ on $H(\mathbb{A}_k)$, on which $Z$ operates via the unitary character $\chi$, generating an irreducible representation with $H_S$-type $\mu_S$ and with*

$$\int_{H(k)Z(\mathbb{A}_k)\backslash H(\mathbb{A}_k)} \tilde{f}\bar{g}\,dh \neq 0.$$

*Proof*

Since $\mu_S$ is a supercuspidal representation of $H_S$, the function $\tilde{f}$ is automatically a cusp form; that is, it satisfies the condition of the vanishing of the integrals on unipotent radicals of proper parabolics. (We note that $\tilde{f}$ may not be an automorphic function, as it may not have the requisite finiteness properties.) This follows because the constant term along the unipotent radical, say, $N$, of a parabolic gives rise to an $N(\mathbb{A}_k)$-invariant linear form on the space obtained by $H(\mathbb{A})$-translates of $\tilde{f}$ and hence must be zero as the $H_S$-translates generate a supercuspidal representation that has no $N_S$-invariant form.

Now, $\tilde{f}$ being a cusp form, it cannot be orthogonal to all irreducible cuspidal automorphic representations; hence there exists an automorphic function $g$ generating an irreducible space of functions with the same central character as $f$ such that

$$\int_{H(k)Z(\mathbb{A}_k)\backslash H(\mathbb{A}_k)} \tilde{f}\bar{g}\,dh \neq 0.$$

The nonvanishing of the integral implies that the $H_S$-type of the space generated by $g$ is the same as that of the space generated by $\tilde{f}$, which is $\mu_S$. □

*Remark 2*

In the proof of Theorem 2, we apply Lemma 1 to G $= \mathrm{GL}_2 \times \mathrm{GL}_2$, $H = \mathrm{GL}_2$, and $Z = G_m$ using the well-known theorem of Kutzko (cf. [K, theorem, page 43]), according to which a supercuspidal representation of $\mathrm{GL}_2(k_u)$ can be obtained as an induced representation from a finite-dimensional representation of an open subgroup which is compact modulo center.

**2. Saito-Tunnell**

By exactly the same method as employed in Section 1, one can deduce the theorem of Saito and Tunnell which describes which characters of $L^*$ for $L$ a quadratic extension of a local field $k_u$ appear in an irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(k_u)$ or in an irreducible representation of $D_u^*$, where $D_u$ is the unique quaternion division algebra over $k_u$, in terms of the local epsilon factors. It is elementary to see that characters of $L^*$ appear with multiplicity at most 1 in any irreducible representation of $\mathrm{GL}_2(k_u)$ or of $D_u^*$ (see, e.g., [P1, Remark 3.5]).

The dichotomy principle also holds in this situation; that is, if $\pi$ is a discrete series representation of $\mathrm{GL}_2(k_u)$ and $\pi'$ is the corresponding finite-dimensional irreducible representation of $D_u^*$, then for a character $\chi$ of $L^*$ whose restriction to $k_u^*$ is the same as the central character of $\pi$, $\chi$ appears in exactly one of the representations $\pi$ or $\pi'$. One can give a proof of this using the character identity

$$\Theta_\pi(x) = -\Theta_{\pi'}(x).$$

By appealing to a global theorem due to Waldspurger, we give a proof of the Saito-Tunnell theorem. However, there seems little point in giving details of the proof except to recall the statement of the theorems.

THEOREM 3 (Waldspurger [W, Theorem 2, page 221])
*Let $k$ be a number field, let $F$ be a quadratic extension of $k$, let $D$ be a quaternion algebra over $k$ containing $F$, and let $\pi'$ be an automorphic representation of $D(\mathbb{A}_k)^*$ realised on a space of functions $E'$ on $D^*\backslash D(\mathbb{A}_k)^*$. Let $T$ be the torus inside $D^*$ defined by $F$, and let $\Omega = \bigotimes \Omega_v$ be a continuous character of $T(k)\backslash T(\mathbb{A}_k)$ whose restriction to the center of $D(\mathbb{A}_k)^*$ is the same as the central character of $\pi'$. Then there exists a function $e'$ in $E'$ such that the integral*

$$\int_{T(k)\mathbb{A}_k^*\backslash T(\mathbb{A}_k)} e'(t)\Omega^{-1}(t)\,dt$$

*is nonzero if and only if the following two conditions are satisfied:*
(1) *for all places $v$ of $k$, the local representation $\pi_v'$ has $\Omega_v$-invariant linear form for the torus $(F \otimes k_v)^*$;*
(2) *if $\Pi$ denotes the base change of $\pi$ to $\mathrm{GL}_2(\mathbb{A}_F)$,*

$$L\Big(\frac{1}{2}, \Pi \otimes \Omega^{-1}\Big) \neq 0.$$

We now state the theorem of Saito [S, theorem, page 99] and Tunnell [T, theorem, page 1277] which follows from Theorem 3 above due to Waldspurger, just as in the proof of Theorem 2. In this theorem of Saito and Tunnell, the epsilon factor is sensitive to the additive character chosen. The theorem is deduced from the theorem of Waldspurger, which has no reference to the additive character. The reason for this, of course, is that the corresponding statement at infinity also depends on the choice of such an additive character. (Note that fixing a character of $\mathbb{A}_k/k$ at one place of $k$ fixes it also at any other place of $k$ because $k_v$ is dense in $\mathbb{A}_k/k$ for any place $v$ of $k$.) The theorem of Saito and Tunnell, although not stated in their articles for Archimedean fields, is valid for such fields too and has quite an elementary proof.

THEOREM 4 (Saito [S, theorem, page 99]; Tunnell [T, theorem, page 1277])
*Let $\pi$ be an infinite-dimensional irreducible admissible representation of* $\mathrm{GL}_2(k_u)$*, let* $L$ *be a quadratic extension of* $k_u$*, and let* $\Pi$ *be the base change lift of* $\pi$ *to* $\mathrm{GL}_2(L)$*. Fix a nontrivial additive character* $\psi$ *of* $L$ *which is trivial on* $k_u$*. Then for a character* $\Omega_u$ *of* $L^*$ *which has the same restriction to* $k_u^*$ *as the central character of* $\pi$*,* $\epsilon(\Pi \otimes \Omega_u^{-1}, \psi)$ *is independent of* $\psi$ *(as long as its restriction to* $k_u$ *is trivial) and takes the value* $\pm 1$*. The character* $\Omega_u$ *of* $L^*$ *appears in* $\pi$ *if and only if*

$$\epsilon(\Pi \otimes \Omega_u^{-1}, \psi) = 1.$$

*Remark 3*
Theorem 4 is proved by applying Lemma 1 to $G = \mathrm{GL}_2$, where $H$ is the 2-dimensional torus defined by $F^*$, and $Z = G_m$.

## 3. Extending Saito-Tunnell

Let $L$ be a quadratic extension of a local field $k_u$ of characteristic $\neq 2$. The theorem of Saito and Tunnell discussed in Section 2 describes the characters of $L^*$ which appear in irreducible admissible infinite-dimensional representations of $\mathrm{GL}_2(k_u)$ or in irreducible representations of $D_u^*$, where $D_u$ is the unique quaternion division algebra over $k_u$ in terms of the local epsilon factors. If the representation $\pi_\theta$ of $\mathrm{GL}_2(k_u)$ comes from a character $\theta$ of $L^*$ via the construction of the Weil representation (see [JL, Theorem 4.6]), the representation $\pi_\theta$ decomposes into two irreducible representations when restricted to $\mathrm{GL}_2(k_u)^+ = \{x \in \mathrm{GL}_2(k_u) \mid \det(x) \in NL^*\}$, where $NL^*$ is the subgroup of $k_u^*$ of index 2 consisting of norms from $L^*$; denote these representations of $\mathrm{GL}_2(k_u)^+$ as $\pi_-$ and $\pi_+$, so that $\pi_\theta = \pi_+ \oplus \pi_-$ on $\mathrm{GL}_2(k_u)^+$. Similarly, one can define a subgroup of index 2 inside $D_u^*$ to be denoted by $D_u^{*+}$ and for which one has a similar decomposition $\pi'_\theta = \pi'_+ \oplus \pi'_-$ of the corresponding representation of $D_u^{*+}$. Clearly, $L^*$ is contained in both $\mathrm{GL}_2(k_u)^+$ and $D_u^{*+}$, and one can ask about a generalisation of the theorem of Saito and Tunnell to describe the decomposition of $\pi_+, \pi_-, \pi'_+, \pi'_-$ restricted to $L^*$. The following is such a theorem. Before stating the theorem, we note that the central character of $\pi_\theta$ is $\theta|_{k_u^*} \cdot \omega_{L/k_u}$.

THEOREM 5
*Let* $\pi_\theta$ *(resp.,* $\pi'_\theta$*) be the irreducible admissible representation of* $\mathrm{GL}_2(k_u)$ *(resp.,* $D_u^*$*) associated to a character* $\theta$ *of* $L^*$*. Fix embeddings of* $L^*$ *in* $\mathrm{GL}_2(k_u)^+$ *and* $D_u^{*+}$*. (In general, there are two conjugacy classes of such embeddings.) Let* $\psi$ *be a nontrivial additive character of* $L$ *trivial on* $k_u$*. The restriction of* $\pi_\theta$ *to* $\mathrm{GL}_2(k_u)^+$ *can be written as* $\pi_\theta = \pi_+ \oplus \pi_-$*, and the restriction of* $\pi'_\theta$ *to* $D_u^{*+}$ *can be written as* $\pi'_\theta = \pi'_+ \oplus \pi'_-$ *such that a character* $\chi$ *of* $L^*$ *with* $(\chi \cdot \theta^{-1})|_{k_u^*} = \omega_{L/k_u}$ *appears in* $\pi_+$ *if and only if* $\epsilon(\theta\chi^{-1}, \psi) = \epsilon(\bar{\theta}\chi^{-1}, \psi) = 1$ *and appears in* $\pi_-$ *if and only if* $\epsilon(\theta\chi^{-1}, \psi) =$

$\epsilon(\bar{\theta}\chi^{-1}, \psi) = -1$. *Similarly, a character* $\chi$ *of* $L^*$ *with* $(\chi \cdot \theta^{-1})|_{k_u^*} = \omega_{L/k_u}$ *appears in* $\pi'_+$ *if and only if* $\epsilon(\theta\chi^{-1}, \psi) = 1$ *and* $\epsilon(\bar{\theta}\chi^{-1}, \psi) = -1$, *and it appears in* $\pi'_-$ *if and only if* $\epsilon(\theta\chi^{-1}, \psi) = -1$ *and* $\epsilon(\bar{\theta}\chi^{-1}, \psi) = 1$.

This theorem was proved in the odd residue characteristic case by this author in [P3]. We now prove this theorem in general by a global argument similar to the one in Section 1. The following lemma plays an important role in transferring information from a finite prime of a number field to an infinite prime.

LEMMA 3
*Let* $k$ *be a complex multiplication (CM) number field, and let* $u$ *be a finite place of* $k$. *Let* $\lambda : k_u^* \to \mathbb{C}^*$ *be a character. Then there exists a Grössencharacter* $\Lambda : \mathbb{A}_k^*/k^* \to \mathbb{C}^*$ *which is unramified at all the finite primes outside* $u$, *has* $\lambda$ *as the local component at* $u$, *and whose local component at infinity is given by*

$$\Lambda_\infty(z_1, \ldots, z_r) = z_1^{n_1} \cdots z_k^{n_r} \quad \text{for } |z_i| = 1,$$

*where* $(n_1, \ldots, n_r)$ *is an element of* $\mathbb{Z}^r$ *with arbitrary values at all but one component.*

*Proof*
Let $U_k = \prod_{v \in S_f} U_v \prod_{v \in S_\infty} \mathbb{S}^1$ be the maximal compact subgroup of $\mathbb{A}_k^*$. Define a character $\mu$ on $U_k$ by declaring its value on $U_u$ to be $\lambda$ restricted to $U_u$, trivial on $U_v$, $v \neq u$, and given on $(z_1, \ldots, z_r) \in (\mathbb{S}^1)^r$ to be $z_1^{n_1} \cdots z_r^{n_r}$. Since $k^* \cap U_k = \mu_k$, the group of roots of unity in $k^*$, $\mu|_{k^* \cap U_k}$, is a character of finite order. Noting that $z_i : \mu_k \to \mathbb{C}^*$ are injective, it follows that $\mu|_{k^* \cap U_k}$ can be assumed to be trivial by changing any one of the components of the $r$-tuple of integers $(n_1, \ldots, n_r) \in \mathbb{Z}^r$.

Thus we have a character, say, $\Lambda'$, of the group $U_k/(k^* \cap U_k)$. Since we have $U_k/(k^* \cap U_k) \hookrightarrow \mathbb{A}_k^*/k^*$ and $U_k/(k^* \cap U_k)$ is a compact group (in particular, closed), the character $\Lambda'$ can be extended to a Grössencharacter, say, $\Lambda''$, of $\mathbb{A}_k^*/k^*$. Observe that, at the moment, we have constructed a Grössencharacter $\Lambda''$ of $\mathbb{A}_k^*/k^*$ which on $U_u$ is $\lambda$ restricted to $U_u$. However, we still have the character $|\cdot| : \mathbb{A}_k^*/k^* \to \mathbb{C}^*$ given by $x \to |x|$ which is trivial on $U_k$ and takes the value $q_u^{-1}$ on $\pi_u \in k_u^* \subset \mathbb{A}_k^*/k^*$. Therefore, for an appropriate choice of $s_0 \in \mathbb{C}$, we can assume that the Grössencharacter $\Lambda = \Lambda''|\cdot|^{s_0}$ has $\lambda$ as its local component at $u$, is unramified at all the other finite places, and has the desired behaviour at infinity. $\square$

*Proof of Theorem 5*
We choose a totally real number field $k$ and a place $u$ of $k$ so that the completion of $k$ at $u$ is $k_u$. Let $F$ be a quadratic extension of $k$ which is a CM-field for which $F \otimes_k k_u \cong L$. Assume that $F$ is split at all the places of $k$ of residue characteristic 2 except $u$ (if $u$ is of residue characteristic 2). Let $D$ be a quaternion division algebra

over $k$ containing $F$ for which $D \otimes_k k_u \cong D_u$, which is split at all the finite places of $k$ except $u$, and which remains a division algebra at all the infinite places of $k$. Such a choice of the triple $(k, F, D)$ is possible, as can be seen; we note that by the conditions on $D$, the degree of $k$ over $\mathbb{Q}$ is odd.

Let $\omega_{F/k} = \prod_v \omega_v$ be the quadratic character of $\mathbb{A}_k^*/k^*$ defining $F$. Define

$$D^{*+}(\mathbb{A}_k) = \left\{ g \in D^*(\mathbb{A}_k),\, g = \prod g_v \,\middle|\, \omega_v\big(\nu_v(g_v)\big) = 1,\ \forall \text{ places } v \text{ of } k \right\},$$

where $\nu_v$ denotes the reduced norm on $D \otimes_k k_v$.

Let $D^{*+}(k) = D^* \cap D^{*+}(\mathbb{A}_k)$. Clearly, $D^{*+}(\mathbb{A}_k)$ is an open subgroup of $D^*(\mathbb{A}_k)$ containing $D^{*+}(k)$.

Let $\pi$ be an automorphic representation of $D^*(\mathbb{A}_k)$ obtained from a character $\Theta : \mathbb{A}_F^*/F^* \to \mathbb{C}^*$ which is unramified at all the finite places of $F$ except $u$ and is $\theta$ on $F_u^*$; this is possible by Lemma 3. Let $\pi^+ = \bigotimes \pi_w^+$ be an irreducible representation of $D^{*+}(\mathbb{A}_k)$ with $\pi_u^+ = \pi_+$ contained in the space of functions on $D^{*+}(\mathbb{A}_k)$ obtained by restricting functions on $D^*(\mathbb{A}_k)$ corresponding to elements of $\pi$.

Let $T$ be the torus inside $D^*$ defined by $F$. By Lemma 1, there exists a continuous character $\Omega_\chi = \bigotimes \Omega_v$ of $T(k)\backslash T(\mathbb{A}_k)$ with $\Omega_u = \chi$ whose restriction to the center of $D(\mathbb{A}_k)^*$ is the same as the central character of $\pi$ and for which there exists a function $e_\chi$ in $\pi^+$ such that the integral

$$\int_{T(k)\mathbb{A}_k^*\backslash T(\mathbb{A}_k)} e_\chi(t)\Omega_\chi^{-1}(t)\, dt$$

is nonzero. By the theorem of Waldspurger, if $\Pi = BC(\pi)$ denotes the base change of $\pi$ to $\mathrm{GL}_2(\mathbb{A}_F)$,

$$L\Big(\frac{1}{2}, \Pi \otimes \Omega_\chi^{-1}\Big) \neq 0.$$

Notice that

$$\Pi = BC(\pi) = Ps(\Theta, \bar{\Theta}),$$

where $\bar{\Theta}$ denotes the conjugate of $\Theta$ under the nontrivial element of the Galois group of $F$ over $k$. It follows that

$$L(s, \Pi \otimes \Omega_\chi^{-1}) = L_F(s, \Theta \cdot \Omega_\chi^{-1}) L_F(s, \bar{\Theta} \cdot \Omega_\chi^{-1}).$$

By the condition on the central characters, one sees that $\mathrm{Ind}_F^k(\Theta \cdot \Omega_\chi^{-1})$ is a self-dual representation, and hence the functional equation for its $L$-function relates it to itself. Therefore, since

$$L_F(s, \Theta \cdot \Omega_\chi^{-1}) = L_k\big(s, \mathrm{Ind}_F^k(\Theta \cdot \Omega_\chi^{-1})\big),$$

the functional equation for $L_F(s, \Theta \cdot \Omega_\chi^{-1})$ relates this $L$-function to itself; similarly, the functional equation for $L_F(s, \bar{\Theta} \cdot \Omega_\chi^{-1})$ relates this $L$-function to itself.

As $L(1/2, \Pi \otimes \Omega_\chi^{-1}) \neq 0$, so also are $L_F(1/2, \Theta \cdot \Omega_\chi^{-1})$ and $L_F(1/2, \bar{\Theta} \cdot \Omega_\chi^{-1})$, and therefore, in particular, the sign in their functional equations is 1:

$$\epsilon_F\left(\frac{1}{2}, \Theta \cdot \Omega_\chi^{-1}\right) = 1,$$

$$\epsilon_F\left(\frac{1}{2}, \bar{\Theta} \cdot \Omega_\chi^{-1}\right) = 1.$$

The global epsilon factor is the product of the local ones:

$$\epsilon_F\left(\frac{1}{2}, \Theta \cdot \Omega_\chi^{-1}\right) = \prod_w \epsilon_w\left(\frac{1}{2}, \Theta_w \cdot \Omega_w^{-1}\right).$$

Using the theorem of Saito and Tunnell, we note that to prove Theorem 5, it suffices to show that for any fixed irreducible component of $\pi_\theta$ restricted to $\mathrm{GL}_2(k_u)^+$, and for characters $\chi$ that occur in this fixed component, both $\epsilon(\theta\chi^{-1}, \psi)$ and $\epsilon(\bar{\theta}\chi^{-1}, \psi)$ are independent of $\chi$ (and similarly for $D_u^{*+}$). Observe that as $\chi$ varies over characters of $L^*$ belonging to the set ! of characters of $L^*$ which appear in the restriction of $\pi_+$ to $L^*$, so also does the character $\Omega_\chi$ with

$$L\left(\frac{1}{2}, \Pi \otimes \Omega_\chi^{-1}\right) \neq 0.$$

Hence, as observed above, $\epsilon_F(1/2, \Theta \cdot \Omega_\chi^{-1}) = 1$, and $\epsilon_F(1/2, \bar{\Theta} \cdot \Omega_\chi^{-1}) = 1$. We prove that for any place $u'$ of $k$ not lying under $u$, $\prod_{w|u'} \epsilon_w(1/2, \theta_w \Omega_w^{-1})$ is independent of these variations in $\Omega_\chi$, and therefore since $\prod \epsilon_w = 1$, $\epsilon_u(1/2, \theta_u \chi^{-1})$ is independent of $\chi$. (As long as it appears in the restriction of $\pi_+$ to $L^*$.) We analyse the local epsilon factors at the various primes of $F$, and thereby prove the theorem, in the following steps.

*Step 1.* For places $u$ of $k$ which split into two places in $F$, say, $w_1$ and $w_2$, we have

$$\epsilon_{w_1}\left(\frac{1}{2}, \Theta_{w_1} \cdot \Omega_{w_1}^{-1}\right) \cdot \epsilon_{w_2}\left(\frac{1}{2}, \Theta_{w_2} \cdot \Omega_{w_2}^{-1}\right) = 1.$$

This follows from the relation

$$\epsilon\left(\frac{1}{2}, \chi, \psi(x)\right) \cdot \epsilon\left(\frac{1}{2}, \chi^{-1}, \psi(-x)\right) = 1,$$

which is what we have by the condition on the central character of $\pi$ (which is $\Theta|_{\mathbb{A}_k^*} \cdot \omega_{F/k}$) being the same as the character $\Omega$ restricted to $\mathbb{A}_k^*$. We also need to note that the additive character of $\mathbb{A}_F$, with respect to which we are calculating that the epsilon factor is trivial on $\mathbb{A}_k$, is of the form $(\psi(x), \psi(-x))$ at a prime $w$ of $k$ which splits in $F$.

*Step 2.* For places $w$ of $F$ which are inert over the corresponding place $v$ in $k$, we note that since, by the choice of $\Omega_\chi$, the $\Omega_\chi$-period integral is nonzero, we have local $\Omega_w$-invariant forms on $\pi_w^+$ at all places. Since $\Theta$ is unramified at all finite places away from $u$ and thus is invariant under $\mathrm{Gal}(F/k)$ at such places, the representation $\pi_w$ is, up to a twist, of the form $Ps(1, \omega)$, where $\omega$ is the quadratic character defining the extension $F_w$ of $k_v$. Lemma 4 now implies that

$$\epsilon_w\Big(\frac{1}{2}, \Theta_w \cdot \Omega_w^{-1}\Big) = \pm 1 \quad \text{but independent of } \Omega_w$$

at all finite places $w$ of $F$ inert over $k$, $w \neq u$.

*Step 3.* The epsilon factor $\epsilon_w(1/2, \Theta_w \cdot \Omega_w^{-1})$ is independent of $\Omega_w$ at any infinite place $w$ as $D$ remains a division algebra at such places. This once again follows because the period integral is nonzero, and hence there is an invariant linear form at every infinite place. Appealing to the Archimedean analogue of our theorem which easily follows from the information given in Section 5, the proof of the theorem is completed. □

LEMMA 4
*Let $L$ be a quadratic extension of a local field $k_u$ of odd residue characteristic. Then the principal series representation $\pi = Ps(1, \omega_{L/k_u})$ splits into two irreducible representations $\pi = \pi_+ \oplus \pi_-$ when restricted to $\mathrm{GL}_2(k_u)^+ = \{x \in \mathrm{GL}_2(k_u) \mid \det(x) \in NL^*\}$, where $NL^*$ is the subgroup of $k_u^*$ of index 2 consisting of norms from $L^*$ such that a character $\chi$ of $L^*$ with $\chi|_{k_u^*} = \omega_{L/k_u}$ appears in $\pi_+$ if and only if $\epsilon(\chi, \psi) = 1$ and appears in $\pi_-$ if and only if $\epsilon(\chi, \psi) = -1$; here, $\psi$ is a nontrivial additive character of $L$ trivial on $k_u$.*

*Proof*
We begin by observing that for a character $\chi$ of $L^*$ with $\chi|_{k_u^*} = \omega_{L/k_u}$, $\epsilon(\chi, \psi) = \pm 1$. This follows from the relation $\epsilon(\chi, \psi)\epsilon(\chi^{-1}, \psi(-x)) = 1$, together with the observation that $\chi^\sigma = \chi^{-1}$, $\psi^\sigma = \psi(-x)$, and $\epsilon(\chi, \psi) = \epsilon(\chi^\sigma, \psi^\sigma)$, where $\sigma$ is the nontrivial element in the Galois group of $L$ over $k_u$.

The following was proved in [P3, Lemma 3.1] by direct calculation in the odd residue characteristic:

$$\epsilon(\omega_{L/k_u}, \psi_0)\frac{\omega_{L/k_u}\Big(\frac{x-\bar{x}}{x_0-\bar{x}_0}\Big)}{\Big|\frac{(x-\bar{x})^2}{x\bar{x}}\Big|_{k_u}^{1/2}} = \sum_{\substack{\epsilon_L(\chi,\psi)=1 \\ \chi|_{k_u^*}=\omega_{L/k_u}}} \chi(x).$$

Here, $\psi_0$ is a nontrivial character of $k$, $x_0$ is an element of $L^*$ whose trace to $k_u$ is zero, and $\psi$ is an additive character on $L$ defined by $\psi_0(x) = \psi(\mathrm{tr}[-xx_0/2])$; as usual, the summation (on the right-hand side) is by partial sums over all characters of $L^*$ of conductor at most $n$.

It has been proved by Langlands [L, Lemma 7.19] (the lemma with an "embarrassing" proof) that the character $\Theta_{\pi_+}$ of $\pi_+$ is given by

$$\Theta_{\pi_+}(x) = \epsilon(\omega_{L/k_u}, \psi_0)\frac{\omega_{L/k_u}\left(\frac{x-\bar{x}}{x_0-\bar{x}_0}\right)}{\left|\frac{(x-\bar{x})^2}{x\bar{x}}\right|_{k_u}^{1/2}}.$$

These two results combine to prove Lemma 4. □

*Remark 4*

We have stated and proved Lemma 4 only in the odd residue characteristic; this is enough for our purposes to prove Theorem 5 in all residue characteristics. The statement of Theorem 5 includes Lemma 4 as a particular case, and therefore, after we have proved Theorem 5 using Lemma 4 in odd residue characteristic, we get a proof of Lemma 4 in *all* residue characteristics by a global method. It is natural to expect that Lemma 4 has more transparent local proof; indeed, this is the case (see the appendix to this article by H. Saito).

Since [L, Lemma 7.19] is available in all residue characteristics, Lemma 4 implies that the following lemma proved by this author only in odd residue characteristic in [P3] is true in all residue characteristics.

LEMMA 5

*Let $L$ be a quadratic extension of a local field $k_u$. Fix a nontrivial character $\psi_0$ of $k_u$, and fix an element $x_0$ in $L^*$ whose trace to $k_u$ is zero. Define an additive character $\psi$ on $L$ by $\psi(x) = \psi_0(\mathrm{tr}[-xx_0/2])$. Then*

$$\epsilon(\omega_{L/k_u}, \psi_0)\frac{\omega_{L/k_u}\left(\frac{x-\bar{x}}{x_0-\bar{x}_0}\right)}{\left|\frac{(x-\bar{x})^2}{x\bar{x}}\right|_{k_u}^{1/2}} = \sum_{\substack{\epsilon(\chi,\psi)=1 \\ \chi|_{k_u^*}=\omega_{L/k_u}}} \chi(x),$$

*where, as usual, the summation on the right-hand side is by partial sums over all characters of $L^*$ of conductor at most $n$.*

*Remark 5*

Lemma 5 says that

$$\epsilon(\omega_{L/k_u}, \psi_0)\frac{\omega_{L/k_u}\left(\frac{x-\bar{x}}{x_0-\bar{x}_0}\right)}{\left|\frac{(x-\bar{x})^2}{x\bar{x}}\right|_{k_u}^{1/2}}$$

has an interpretation via epsilon factors. This function is, of course, the famous transfer factor that first made its appearance in the work of Labesse and Langlands [LL]. It is not inconceivable that transfer factors in general are related to epsilon factors in a similar way.

*Remark 6*

Given a quadratic extension $L$ of a local field $k_u$, the group $L^1$ of norm 1 elements of $L^*$ can be embedded in $\mathrm{SL}_2(k_u)$. The twofold metaplectic cover $Mp_2(k_u)$ of $\mathrm{SL}_2(k_u)$ splits over $L^1$. (However, the splitting depends on the choice of a character $\chi_0$ of $L^*$ such that $\chi_0|k_u^* = \omega_{L/k_u}$.) Thus one can speak of the restriction of the Weil representation $\omega_\psi$ of $Mp_2(k_u)$, associated to a nontrivial additive character $\psi$ of $k_u$, to $L^1$. Characters $\mu$ of $L^1$ can be identified to characters $\chi$ of $L^*$ with $\chi|k_u^* = \omega_{L/k_u}$ by $\chi(x) = \mu(x/\bar{x})\chi_0(x)$. It is a theorem of Moen [M], in odd residue characteristic, proved by Rogawski [Ro], in general, that a character $\chi$ of $L^*$ with $\chi|k_u^* = \omega_{L/k_u}$ appears in $\omega_\psi$ if and only if $\epsilon(\chi, \psi) = 1$. Thus Lemma 4 implies that the Weil representation $\omega_\psi$ restricted to $L^1$ is closely related to a component of a reducible principal series. What lies behind this phenomenon-relating character of a linear and a nonlinear group is not clear to this author.

## 4. Symmetric and exterior squares

The results in [P1] on the trilinear forms were refined in [P4] in the case when $\pi_1 = \pi_2 = \pi$. In this case,

$$\pi \otimes \pi = \mathrm{Sym}^2(\pi) \oplus \bigwedge^2(\pi).$$

Similarly, for the corresponding Galois representations,

$$\sigma_\pi \otimes \sigma_\pi = \mathrm{Sym}^2(\sigma_\pi) \oplus \bigwedge^2(\sigma_\pi).$$

Following is the theorem about symmetric squares proved in [P4] in the odd residue characteristic and now proved in general. It should also be pointed out that the methods of the Weil representation employed in [P4] could prove results about symmetric and exterior squares only for $\mathrm{GL}_2$ and not for the representations of invertible elements of a division algebra.

THEOREM 6

*Let $D_u$ be a quaternion algebra over $k_u$. Let $\epsilon_D$ be 1 if $D_u = M_2(k_u)$, and let $\epsilon_D = -1$ otherwise. For irreducible admissible representations $\pi$ and $\pi'$ of $D_u^*$ (which are assumed to be infinite-dimensional if $D_u^* = \mathrm{GL}_2(k_u)$) with $\omega_\pi^2\omega_{\pi'} = 1$, $\mathrm{Sym}^2(\pi) \otimes \pi'$ has a $D_u^*$-invariant linear form if and only if $\epsilon(\mathrm{Sym}^2(\sigma_\pi) \otimes \sigma_{\pi'}) =$*

$\omega_\pi(-1)$, *and* $\epsilon\big(\bigwedge^2(\sigma_\pi)\otimes\sigma_{\pi'}\big) = \epsilon_D\omega_\pi(-1)$. *The representation* $\bigwedge^2(\pi)\otimes\pi'$ *has a* $\mathrm{GL}_2(k_u)$*-invariant linear form if and only if* $\epsilon(\mathrm{Sym}^2(\sigma_\pi)\otimes\sigma_{\pi'}) = -\omega_\pi(-1)$, *and* $\epsilon\big(\bigwedge^2(\sigma_\pi)\otimes\sigma_{\pi'}\big) = -\epsilon_D\omega_\pi(-1)$. *(The epsilon factors in this theorem are independent of the additive character of* $k_u$*, which hence is omitted from the notation.)*

*Proof*

Let $k$ be a totally real number field with a finite place $u$, where the completion of $k$ is $k_u$. Let $D$ be a quaternion division algebra over $k$ for which $D\otimes_k k_u \cong D_u$, which is split at all the other finite primes and which remains a division algebra at all the places at infinity. Such a choice of the pair $(k, D)$ exists.

Let $\Pi$ be an irreducible automorphic representation of $D^*(\mathbb{A}_k)$ with $\pi$ as the local component at $u$, unramified at all the other finite places of $k$, and certain representations at infinity. By Lemma 1, there exists a representation $\Pi'$ of $D^*(\mathbb{A}_k)$ with local component $\pi'$ at $u$, functions $f_1, f_2 \in \Pi$, and $f' \in \Pi'$ such that

$$\int_{D^*\mathbb{A}_k^*\backslash D^*(\mathbb{A}_k)} f_1 f_2 f'\, dg \neq 0.$$

By the theorem of Harris and Kudla (see [HK]), $L(1/2, \Pi\times\Pi\times\Pi') \neq 0$. The proof proceeds, once again, using the factorisation of $L$-functions,

$$L(s, \Pi\times\Pi\times\Pi') = L\big(s, \mathrm{Sym}^2(\Pi)\times\Pi'\big)L\Big(s, \bigwedge^2(\Pi)\times\Pi'\Big),$$

all of which are known to be analytic at $1/2$. Therefore, if $L(1/2, \Pi\times\Pi\times\Pi') \neq 0$, then both $L(1/2, \mathrm{Sym}^2(\Pi)\times\Pi')$ and $L\big(1/2, \bigwedge^2(\Pi)\times\Pi'\big)$ are nonzero. Since we are once again dealing with self-dual representations, this forces global epsilon factors to be 1:

$$\epsilon\Big(\frac{1}{2}, \mathrm{Sym}^2(\Pi)\times\Pi'\Big) = 1,$$

$$\epsilon\Big(\frac{1}{2}, \bigwedge^2(\Pi)\times\Pi'\Big) = 1.$$

We have the factorisation of epsilon factors

$$\epsilon\Big(\frac{1}{2}, \mathrm{Sym}^2(\Pi)\times\Pi'\Big) = \prod_v \epsilon\Big(\frac{1}{2}, \mathrm{Sym}^2(\Pi_v)\times\Pi'_v\Big).$$

Observe that the linear form

$$f_1\otimes f_2\otimes f' \longrightarrow \int_{D^*\mathbb{A}_k^*\backslash D^*(\mathbb{A}_k)} f_1 f_2 f'\, dg$$

defines a $D^*(\mathbb{A}_k)$-invariant linear form on $\Pi \otimes \Pi \otimes \Pi'$ which is symmetric in the first two variables. By the local uniqueness of the trilinear form, the invariant form on $\Pi_w \otimes \Pi_w \otimes \Pi'_w$ is either symmetric or skew-symmetric in the first two variables. By generalities about group representations, it follows that the set of places $w$ of $k$ for which the invariant form on $\Pi_w \otimes \Pi_w \otimes \Pi'_w$ is skew-symmetric is even. By choice, $\Pi_w$ is a principal series at all finite places $w \neq u$. By the following lemma, the local invariant forms on $\Pi_w \otimes \Pi_w \otimes \Pi'_w$ at finite places $w \neq u$ are symmetric or skew in the first two variables, depending on whether $\epsilon(\omega_{\pi_v} \cdot \pi'_v) = \omega_{\pi_v}(-1)$ or $\epsilon(\omega_{\pi_v} \cdot \pi'_v) = -\omega_{\pi_v}(-1)$.

LEMMA 6
*Suppose that $\pi_v$ is a principal series representation of $\mathrm{GL}_2(k_v)$. Then for an irreducible admissible representation $\pi'_v$ of $\mathrm{GL}_2(k_v)$, $\mathrm{Sym}^2(\pi_v) \otimes \pi'_v$ has a $\mathrm{GL}_2(k_v)$-invariant linear form if and only if $\epsilon(\omega_{\pi_v} \cdot \pi'_v) = \omega_{\pi_v}(-1)$.*

From the explicit realisation of the principal series, symmetric square too can be explicitly realised. The proof of Lemma 6 eventually boils down to the following, which is easy to see.

LEMMA 7
*For an irreducible representation $\pi$ of $\mathrm{GL}_2(k_v)$ with $\omega_\pi = \chi^2$, let $\ell : \pi \to \mathbb{C}$ be the unique linear form on which the diagonal subgroup $T$ consisting of $(x, y) \in k^* \times k^*$ operates by $\chi(xy)$. Then $\ell$ is left invariant by $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if and only if $\epsilon(\pi \otimes \chi^{-1}) = \chi(-1)$.*

Theorem 6 now follows from the following lemma at infinity; its simple proof is omitted.

LEMMA 8
*Let $\mathbb{H}$ be the quaternion division algebra over $\mathbb{R}$. For irreducible representations $\pi$ and $\pi'$ of $\mathbb{H}^*$ with $\omega_\pi^2 \omega_{\pi'} = 1$, $\mathrm{Sym}^2(\pi) \otimes \pi'$ has an $\mathbb{H}^*$-invariant linear form if and only if $\epsilon(\mathrm{Sym}^2(\sigma_\pi) \otimes \sigma_{\pi'}) = \omega_\pi(-1)$ and $\epsilon\left(\bigwedge^2(\sigma_\pi) \otimes \sigma_{\pi'}\right) = -\omega_\pi(-1)$. The representation $\bigwedge^2(\pi) \otimes \pi'$ has an $\mathbb{H}^*$-invariant linear form if and only if $\epsilon(\mathrm{Sym}^2(\sigma_\pi) \otimes \sigma_{\pi'}) = -\omega_\pi(-1)$ and $\epsilon\left(\bigwedge^2(\sigma_\pi) \otimes \sigma_{\pi'}\right) = \omega_\pi(-1)$.*

This completes the proof of Theorem 6. □

## 5. Archimedean case

The results in this article have eventually depended on similar but much simpler questions for the Archimedean field (which can be assumed to be $\mathbb{R}$). For this, we fix some notation and recall some standard facts without going into the proofs of the

Archimedean lemmas used in this article. We let $\mathbb{H}$ denote the quaternion division algebra over $\mathbb{R}$.

The Weil group $W_{\mathbb{C}/\mathbb{R}}$ of $\mathbb{R}$ is the normaliser of $\mathbb{C}^*$ in $\mathbb{H}^*$ and sits in the exact sequence

$$0 \to \mathbb{C}^* \to W_{\mathbb{C}/\mathbb{R}} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Let $\psi(x) = \exp(2\pi i x)$ be the character of $\mathbb{R}$, and let $\psi_0(z) = \psi(\mathrm{tr}_{\mathbb{C}/\mathbb{R}}\{iz\})$ be a character of $\mathbb{C}$ which is trivial on $\mathbb{R}$. Then for integers $n, m$ and complex number $s$, we have

$$\begin{aligned} \epsilon(z^n \bar{z}^m |z|^s, \psi_0) &= (-1)^{n-m} \quad (\text{if } n \geq m) \\ &= 1 \quad (\text{if } n < m). \end{aligned}$$

For $m \geq 0$, let $\sigma_m$ be the 2-dimensional representation $\mathrm{Ind}_{\mathbb{C}^*}^{W_{\mathbb{C}/\mathbb{R}}} (z/|z|)^m$ of $W_{\mathbb{C}/\mathbb{R}}$. We have

$$\epsilon(\sigma_m, \psi) = i^{m+1} \quad (\text{for } m \geq 0).$$

It follows that

$$\begin{aligned} \epsilon(\sigma_m \otimes \sigma_n, \psi) &= (-1)^{m+1} \quad (\text{for } m \geq n) \\ &= (-1)^{n+1} \quad (\text{for } n \geq m). \end{aligned}$$

Under the Langlands correspondence, the discrete series representation $D_m$ of $\mathrm{GL}_2(\mathbb{R})$ for $m \geq 2$, which has trivial central character restricted to $\mathbb{R}^{*+}$, corresponds to the representation $\sigma_{m-1}$ of the Weil group $W_{\mathbb{C}/\mathbb{R}}$. The corresponding representation of $\mathbb{H}^*/\mathbb{R}^{*+}$, denoted by $F_{m-2}$, is of dimension $m-1$ and of highest weight $z^{m-2}/|z|^{m-2}$.

The proof of the following lemma is standard and is therefore omitted. Using the information on epsilon factors given in this section, this is then easily seen to be equivalent to Lemma 8.

LEMMA 9

*Let $F_n$ denote the irreducible representation of $\mathbb{H}^*/\mathbb{R}^{*+}$ of highest weight $n$. Then*

$$\begin{aligned} \mathrm{Sym}^2(F_n) &= F_{2n} \oplus F_{2n-4} \oplus \cdots, \\ \Lambda^2 F_n &= F_{2n-2} \oplus F_{2n-6} \oplus \cdots. \end{aligned}$$

*Remark 7*

For the purposes of this article, it is curious to note that we can assume that the groups are compact at infinity, and therefore we can deduce theorems about (discrete series representations of) a noncompact real group from that of its compact form via the global theorems. In fact, we expect the same to hold for the Archimedean branching laws from $\mathrm{SO}(p, q)$ to $\mathrm{SO}(p, q-1)$ as conjectured in [GP1], which has proved to be quite difficult to handle by local means.

## 6. Some remarks on the Gross-Prasad conjecture

Let $V$ be a finite-dimensional vector space over a local field $k$, say, of dimension $n > 1$, together with a nondegenerate quadratic form $q$. Let $W$ be a nondegenerate subspace of $(V, q)$ of codimension 1. In this section, we reformulate a weaker version of the Gross-Prasad conjecture from [GP1, Conjecture 10.7] about restriction of representations of $\mathrm{SO}(V)$ to $\mathrm{SO}(W)$, as suggested by the arguments in this article, which we elaborate upon later in the section.

Much of the conjecture in [GP1] is based on having a parametrisation of representations inside an $L$-packet for orthogonal groups. For Archimedean fields, such a parametrisation has been known for a long time, but perhaps this is not going to happen soon for non-Archimedean fields. However, by the work of Jiang and Soudry [JS1], [JS2], there is now a concept of Langlands parameter for all (generic) representations of orthogonal groups (except that their work is at the moment only for split odd orthogonal groups); we assume that their work about Langlands parametrisation has been extended for all quasi-split orthogonal groups.

*Remark 8*

The theorem of Jiang and Soudry is especially pleasant to state for generic discrete series representations of odd orthogonal groups, where it asserts (cf. [JS2, Theorem 2.2]) that there exists a bijective correspondence between irreducible generic discrete series representations of $\mathrm{SO}_{2m+1}(k)$ and irreducible generic representations of $\mathrm{GL}_{2m}(k)$ with Langlands parameter of the form

$$\sigma = \sum \sigma_i,$$

where $\sigma_i$ are irreducible symplectic representations of $W_k'$ which are pairwise distinct (so $\sigma$ is a multiplicity-free sum of irreducible representations). In this form, the statement for even orthogonal groups just needs a change from *symplectic* to *orthogonal* with a condition on the determinant of the full representation.

The work of Jiang and Soudry deals only with generic representations. To circumvent this problem, we need to introduce the concept of *near-equivalence* for representations of groups over local fields introduced via global means as a substitute for the conjectural definition of $L$-packet, presumably defined locally. The concept of near-equivalence which we review now is due to Piatetski-Shapiro. Let $V$ and $V'$ be two quadratic spaces of the same dimension and the same discriminant over a local field $k$. An irreducible admissible representation $\pi$ of $\mathrm{SO}(V)$ is said to be nearly equivalent to a representation $\pi'$ of $\mathrm{SO}(V')$ if there exists a number field $F$ together with quadratic spaces $V_F$ and $V'_F$ of the same discriminant such that the completion of $F$ at some place, say, $v_0$, is $k$; and $V_F \otimes_F k = V$, $V'_F \otimes_F k = V'$ as quadratic spaces; and such that there exist automorphic representations $\Pi = \bigotimes \Pi_v$, $\Pi' = \bigotimes \Pi'_v$ of $\mathrm{SO}(V_F)$ and $\mathrm{SO}(V'_F)$ such that $\Pi_v = \Pi'_v$ for almost all places $v$ of $F$ (note that $\mathrm{SO}(V_F \otimes F_v) \cong \mathrm{SO}(V'_F \otimes F_v)$ for almost all places $v$ of $F$) and that $\Pi_{v_0} = \pi$ and $\Pi'_{v_0} = \pi'$. The concept of near-equivalence makes sense, in particular, if $V = V'$. It is reasonable to expect that if $V = V'$, then the near-equivalence classes of tempered representations are nothing but the set of tempered $L$-packets for $\mathrm{SO}(V)$ and that with $V$ given, as $V'$ varies with $\mathrm{disc}(V) = \mathrm{disc}(V')$, the set of representations $\pi'$ of $\mathrm{SO}(V')$ nearly equivalent to $\pi$ on $\mathrm{SO}(V)$ is what is called a Vogan $L$-packet (see [GP1]). In particular, we *declare* that nearly equivalent (tempered) representations have the same Langlands parameter.

Assuming that every tempered representation of $\mathrm{SO}(V)$ is nearly equivalent to a unique generic representation as conjectured by Shahidi, we get the notion of a Langlands parameter for *any* tempered representation from the theorem of Jiang and Soudry (see [JS1], [JS2]) extended to even orthogonal groups.

The following is a variation on the Gross-Prasad conjecture, [GP1, Conjecture 10.7].

CONJECTURE 1

(1) *Let $\pi$ be an irreducible admissible tempered representation of* $\mathrm{SO}(V)$. *Let the Langlands parameter for $\pi$ be $\sigma = n_1\sigma_1 \oplus n_2\sigma_2 \oplus \cdots \oplus n_r\sigma_r \oplus \tau$, where $\sigma_i$ are the distinct irreducible self-dual representations of the Weil-Deligne group appearing in $\sigma$ which are orthogonal if* $\dim V$ *is even and symplectic if* $\dim V$ *is odd. Suppose that an irreducible admissible representation $\pi'$ of* $\mathrm{SO}(W)$ *appears in $\pi$ as a quotient with Langlands parameter $\sigma'$. Then the epsilon factors $\epsilon(\sigma_i \otimes \sigma')$ are independent of $\pi'$.*

(2) *Part (1) gives a character on $(\mathbb{Z}/2)^r$. If two representations $\pi_1$ and $\pi_2$ of* $\mathrm{SO}(V_1)$ *and* $\mathrm{SO}(V_2)$ *are nearly equivalent, then the corresponding characters are the same if and only if $V_1 = V_2$ and $\pi_1 = \pi_2$.*

(3) *Given a Langlands parameter $\sigma = n_1\sigma_1 \oplus n_2\sigma_2 \oplus \cdots \oplus n_r\sigma_r \oplus \tau$, every character of $(\mathbb{Z}/2)^r$ arises through the construction in (1), that is, for some $\pi$ on some* $\mathrm{SO}(V)$ *with Langlands parameter $\sigma$.*

(4) *One can interchange the roles of $\pi$ and $\pi'$ and make an independence statement for the epsilon factors $\epsilon(\sigma \otimes \sigma'_i)$ in which $\pi$ is varying with Langlands parameter $\sigma$ and $\pi'$ is fixed (with $\pi'$ a quotient of $\pi$) with Langlands parameter $\sigma' = n_1\sigma'_1 \oplus n_2\sigma'_2 \oplus \cdots \oplus n_r\sigma'_r \oplus \tau'$, as before.*

*Remark 9*

In the conjecture of Langlands and Vogan as given in [GP1], one needs to have for each character of the component group a representation of an inner form of the group. By parts (2) and (3) of Conjecture 1, one can in fact use the epsilon factors to serve this purpose; that is, the internal structure of an $L$-packet is dictated by certain epsilon factors.

We now recall the following global conjecture from [GP1, Conjecture 14.8]. It has been suggested by D. Ginzburg in discussions with this author that except for some minor hitch that should be possible to overcome soon, the method of [GJR] is general, and it should prove Conjecture 1 and its generalisation in [GP2].

CONJECTURE 2

*Let $W$ be a nondegenerate codimension $1$ subspace of a finite-dimensional quadratic space $(V, q)$ over a number field $F$. Assume that $\mathrm{SO}(V)$ and $\mathrm{SO}(W)$ are quasi-split, that $\Pi_V$ is an automorphic form on $\mathrm{SO}(V)$, and that $\Pi_W$ is an automorphic form on $\mathrm{SO}(W)$, with both assumed to be generic and tempered. Then there exist a pair $(V', W')$ of a quadratic space $V'$ over $F$, a subspace $W'$ of $V'$ of codimension $1$ with the same dimension and discriminant as that of $(V, W)$, automorphic forms $\Pi'_V$, $\Pi'_W$ on $\mathrm{SO}(V')$, $\mathrm{SO}(W')$ nearly equivalent to $\Pi_V$, $\Pi_W$ (i.e., equal at almost all places of $F$), and functions $f \in \Pi'_V$, $g \in \Pi'_W$ such that the period integral*

$$\int_{\mathrm{SO}(W')\backslash \mathrm{SO}(W')(\mathbb{A})} f g \, d\mu \neq 0$$

*if and only if $L(1/2, \Pi \times \Pi') \neq 0$.*

The suggested proof of Conjecture 1 for discrete series representations depends on globalising a local representation without introducing any ramification at other finite primes, keeping the global orthogonal group in fact compact at infinite places. Further, one assumes that the global parameter, or the corresponding global representation, say, $\Pi$, of $\mathrm{GL}_{2n}$ is an isobaric sum of $r$ self-dual cuspidal representations

$$\Pi = \Pi_1 \oplus \cdots \oplus \Pi_r,$$

reflecting the local decomposition $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_r$. Using Lemma 1 (the Burger-Sarnak principle), one can create $\Pi'$ with local component $\pi'$ and with nonvanishing global $L$-value $L(1/2, \Pi \otimes \Pi')$. The nonvanishing of the global $L$-value $L(1/2, \Pi \otimes \Pi')$ once again implies nonvanishing of $L(1/2, \Pi_i \otimes \Pi')$ and hence forces the global sign in the functional equation $\epsilon(1/2, \Pi_i \otimes \Pi') = 1$. Now, information at unramified primes and at infinity, where the group is compact, can be used to deduce Conjecture 1.

Thus what seems to be necessary to complete this line of argument is used to prove Conjecture 1(1) for unramified tempered representations of $\mathrm{SO}(V)$, that is, to prove that $\epsilon_v(1/2, \Pi_{i,v} \otimes \Pi'_v)$ is independent of the representation $\Pi'_v$ as long as it appears as a quotient of $\Pi_v$. We have not managed to prove this. Especially for a unitary principal series that is not irreducible, we have to consider an analogue of Lemma 4, which seems unclear at the moment.

*Remark 10*

The global methods used in this article, such as in the proof of Theorem 2, have the flavor of results that say that if $\pi'$ appears in $\pi$ as a quotient, then something happens. Thus global methods say nothing if $\pi'$ does not appear as a quotient in $\pi$. In the earlier parts of the work dealing with $\mathrm{GL}_2$, dichotomy was used to force $\pi'$ to appear as a quotient in $\pi$ (either on $\mathrm{GL}_2$ or else on $D^*$). As we mentioned in the introduction, dichotomy is a reflection of certain character identities generalising the character identity of Jacquet and Langlands between inner forms and should hold here too, giving a proof of a generalised form of the dichotomy principle that is [GP1, Conjecture 8.6] as, for instance, in [P2, Proposition 4.3.1], which essentially uses only the character identity of Jacquet and Langlands.

*Remark 11*

We note that the multiplicity one property of restriction of an irreducible representation of $\mathrm{SO}(V)$ to $\mathrm{SO}(W)$ (which although announced by Bernstein and Rallis has not been published) can easily be seen to imply the multiplicity one property for restriction of an irreducible representation of $O(V)$ to $O(W)$, where $O(W)$ sits inside $O(V)$ by acting as 1 on $W^{\perp}$, a line in $V$. The Gross-Prasad conjecture can in fact be formulated for the pair $(O(V), O(W))$ and thus give finer information than the original conjecture, such as the information on symmetric and exterior squares of representations of $\mathrm{GL}_2$ studied in Section 4, rather than just the information about tensor products which is what the original conjectures of [GP1] amount to in this case.

## 7. Epilogue

Results in this article were proved using the Burger-Sarnak principle after applying either the theorem of Harris and Kudla or of Waldspurger, relating the period integrals to central critical $L$-values. These results enabled us to prove nonvanishing of central critical value of certain $L$-functions which played a crucial role in our applications. There is a very large literature on the nonvanishing results of this kind, but we do not detail them here. However, we take the opportunity to state the following general questions about $L$-functions which we have answered in this article in some of the cases we studied here and which may be amenable by similar methods (if the nonvanishing is related to some period integral as conjectured in [GP1] and [GP2] covering the case of two self-dual representations with symplectic parameter for tensor product).

*Question 1*
Given a cuspidal automorphic representation $\Pi$ of $\mathrm{GL}_n(\mathbb{A}_k)$ and an integer $m \le n$, is there a cuspidal automorphic representation $\Pi'$ on $\mathrm{GL}_m(\mathbb{A}_k)$ with prescribed discrete series behaviour at finitely many places of $k$ and with trivial central character for $\Pi \times \Pi'$ such that

$$L\Big(\frac{1}{2}, \Pi \times \Pi'\Big) \neq 0?$$

*Question 2*
Given a cuspidal automorphic representation $\Pi$ of $\mathrm{GL}_n(\mathbb{A}_k)$ which is self-dual up to a twist and an integer $m \le n$, is there a cuspidal automorphic representation $\Pi'$ on $\mathrm{GL}_m(\mathbb{A}_k)$ which is self-dual up to a twist and has prescribed discrete series behaviour at finitely many places of $k$ such that $\Pi \times \Pi'$ is self-dual and such that

$$L\Big(\frac{1}{2}, \Pi \times \Pi'\Big) \neq 0?$$

## Appendix. A local proof of Lemma 4

HIROSHI SAITO

In this appendix, we give a local proof of Lemma 4. (The proof given by Dipendra Prasad is local in odd residue characteristic using explicit calculations with the epsilon factors and a lemma of Langlands [L, Lemma 7.19] but then uses global methods to complete the proof in all residue characteristics.) The proof here follows that of [HKS, Corollary 8.3] and is based on the intertwining operators of principal series representations and the local functional equations of characters of local fields.

Let $F$ be a non-Archimedean local field of characteristic not equal to 2, and let $L$ be a quadratic extension of $F$. Let $O = O_F$ be the ring of integers of $F$, let $P_F$ be its maximal ideal, and let $q = |O/P_F|$. We fix a prime element $\varpi$ of $F$. Let $\omega_{L/F}$ be the character of $F^\times$ corresponding to $L/F$. We fix $x_0 \in L^\times$, whose trace to $F$ is zero, and set $t_0 = x_0^2$. We consider $L^\times$ as a subgroup of $\mathrm{GL}_2(F)$ by

$$a + bx_0 \longmapsto \begin{pmatrix} a & bt_0 \\ b & a \end{pmatrix}.$$

Let $B$ be the subgroup of $\mathrm{GL}_2(F)$ consisting of upper triangular matrices. Then it is easy to check that $\mathrm{GL}_2(F) = BL^\times$ or, explicitly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{d^2 - t_0c^2} \begin{pmatrix} ad - bc & -act_0 + bd \\ 0 & d^2 - t_0c^2 \end{pmatrix} \begin{pmatrix} d & ct_0 \\ c & d \end{pmatrix}.$$

We note that $B \cap L^\times = Z$ is the center of $\mathrm{GL}_2(F)$.

For two quasicharacters $\omega_1, \omega_2$ of $F^\times$ with $\omega_1 \neq \omega_2$, let $Ps(\omega_1, \omega_2)$ be the normalised principal series representation associated with $(\omega_1, \omega_2)$. For a quasicharacter $\chi$ of $L^\times$ satisfying $\chi|_{F^\times} = \omega_1\omega_2$, we define a function $f_{\chi,\omega_1,\omega_2}$ on $\mathrm{GL}_2(F)$ as follows. Let

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} x, \quad x \in L^\times,$$

and set

$$f_{\chi,\omega_1,\omega_2}(g) = \left|\frac{a}{d}\right|^{1/2} \omega_1(a)\omega_2(d)\chi(x).$$

Then by the condition on $\chi$, $f_\chi$ is well defined, and it is easy to see that $f_{\chi,\omega_1,\omega_2}$ is contained in the space $Ps(\omega_1, \omega_2)$. Let $\rho_{\omega_1,\omega_2}$ be the action of $\mathrm{GL}_2(F)$ on $Ps(\omega_1, \omega_2)$. Then $f_{\chi,\omega_1,\omega_2}$ satisfies

$$\rho_{\omega_1,\omega_2}(x) f_{\chi,\omega_1,\omega_2} = \chi(x) f_{\chi,\omega_1,\omega_2}, \quad x \in L^\times.$$

The $\chi$-eigensubspace for $L^\times$ of $Ps(\omega_1, \omega_2)$ is spanned by $f_{\chi,\omega_1,\omega_2}$ (e.g., see [H]).

We define an operator $T\colon Ps(1, \omega_{L/F}) \longrightarrow Ps(1, \omega_{L/F})$ of order 2. Let $T_{\omega_{L/F}}$ be the normalised intertwining operator

$$T_{\omega_{L/F}}\colon Ps(1, \omega_{L/F}) \longrightarrow Ps(\omega_{L/F}, 1)$$

given by

$$(T_{\omega_{L/F}} f)(g) = \gamma(1, \omega_{L/F}, \psi_0)^{-1} \int_F f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g\right) da.$$

Here, $\psi_0$ is a nontrivial additive character of $F$, and $da$ is the self-dual measure of $F$ for $\psi_0$. The function $\gamma(s, \omega_{L/F}, \psi_0)$ is the local gamma factor of the local functional equation of $\omega_{L/F}$. The integral is defined by the analytic continuation of the same integral defining the intertwining operator from $Ps(|\ |^s, \omega_{L/F})$ to $Ps(\omega_{L/F}, |\ |^s)$ for $\Re(s) > 0$ (see [B, Propositions 3.1.5, 4.5.7, 4.5.10]). For a function $f$ on $\mathrm{GL}_2(F)$, we define

$$(I_{\omega_{L/F}} f)(g) = \omega_{L/F}(\det g) f(g).$$

Then $I_{\omega_{L/F}}$ induces a map from $Ps(1, \omega_{L/F})$ to $Ps(\omega_{L/F}, 1)$ and also one from $Ps(\omega_{L/F}, 1)$ to $Ps(1, \omega_{L/F})$. We define $T = I_{\omega_{L/F}} T_{\omega_{L/F}}$. Then $T$ induces a map from $Ps(1, \omega_{L/F})$ to itself satisfying

$$T\rho_{1,\omega_{L/F}}(g) = \omega_{L/F}(\det g)\rho_{1,\omega_{L/F}}(g)T.$$

Let $T'_{\omega_{L/F}}$ be the normalised intertwining operator from $Ps(\omega_{L/F}, 1)$ to $Ps(1, \omega_{L/F})$. Then $T_{\omega_{L/F}} I_{\omega_{L/F}} = I_{\omega_{L/F}} T'_{\omega_{L/F}}$. We see that $T^2$ is the identity since $T'_{\omega_{L/F}} T_{\omega_{L/F}}$ is the identity.

Let

$$Ps(1, \omega_{L/F})^{\pm 1} = \left\{ f \in Ps(1, \omega_{L/F}) \,\middle|\, Tf = \pm f \right\}.$$

Then the two subspaces $Ps(1, \omega_{L/F})^{\pm}$ of $Ps(1, \omega_{L/F})$ are stable under $\mathrm{GL}_2(F)^+ = \{\, g \in \mathrm{GL}_2(F) \mid \omega_{L/F}(\det g) = 1 \,\}$, and

$$Ps(1, \omega_{L/F}) = Ps(1, \omega_{L/F})^+ \oplus Ps(1, \omega_{L/F})^-$$

as $\mathrm{GL}_2(F)^+$-modules. By the multiplicity one property of the action of $L^\times$ on $Ps(1, \omega_{L/F})$, we have $Tf_\chi = c_\chi f_\chi$ for $f_\chi = f_{\chi,1,\omega_{L/F}}$ with $c_\chi = \pm 1$. Obviously, the following result on $c_\chi$ implies Lemma 4.

THEOREM

*Let $\psi(x) = \psi_0(\mathrm{tr}[-xx_0/2])$ for $x \in L$. For a character $\chi$ of $L^\times$ with $\chi|_{F^\times} = \omega_{L/F}$, one has*

$$c_\chi = \epsilon(\chi, \psi).$$

*Proof*

We note that $|d^2 - t_0 c^2|$ takes a finite number of values on $\mathrm{GL}_2(O)$. By dividing $\mathrm{GL}_2(O)$ into a finite number of open sets according to the value of $|d^2 - t_0 c^2|$, we easily see that $f_{\chi,s}(g) = f_{\chi,|\ |^s,\omega_{L/F}}$ is a finite linear combination of flat sections of the family of the representations $Ps(|\ |^s, \omega_{L/F})$ with coefficients in holomorphic functions in $s$. Therefore, $c_\chi = (Tf_\chi)(1)$ is the value at $s = 0$ of the function $\gamma(1, \omega_{L/F}, \psi_0)^{-1} I(s)$ with

$$I(s) = \int_F |a^2 - t_0|^{-1/2-s} \chi(a + x_0)\, da.$$

The integral $I(s)$ converges absolutely for $\Re(s) > 0$ and can be continued to the whole plane as a holomorphic function.

We recall [HKS, Proposition 8.2] in our notation. It says that

$$I(s) = \chi(2x_0)|4^2 t_0|^{-s} \frac{\gamma(s+1/2, \chi^{-1}, \psi_L)}{\gamma(2s, \omega_{L/F}^{-1}, \psi)}.$$

Here, $\psi_L(x) = \psi(\mathrm{tr}(x))$,

$$\gamma(s, \chi^{-1}, \psi_L) = \epsilon(s, \chi^{-1}, \psi_L) \frac{L(1-s, \chi)}{L(s, \chi^{-1})},$$

and $\gamma(s, \omega_{L/F}, \psi)$ is given by a similar formula. We note that $\chi(\varpi) = \omega_{L/F}(\varpi) = -1$ if $L/F$ is unramified, and $\chi^{-1}(x) = \chi(\bar{x})$ for $x \in L^\times$ since $\chi|_{F^\times} = \omega_{L/F}$. Hence we have

$$\begin{aligned} \chi(2x_0)\gamma\left(\frac{1}{2}, \chi^{-1}, \psi_L\right) &= \chi(2x_0)\epsilon(\chi^{-1}, \psi_L) \\ &= \chi(2x_0)\epsilon(\chi, \psi_L) \\ &= \chi(2x_0)\chi\left(\frac{-x_0}{2}\right)^{-1}\epsilon(\chi, \psi_0) \\ &= \chi(-1)\epsilon(\chi, \psi_0). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \gamma(1, \omega_{L/F}, \psi)\gamma(0, \omega_{L/F}, \psi) &= \epsilon(1, \omega_{L/F})\epsilon(0, \omega_{L/F}, \psi) \\ &= \epsilon(\omega_{L/F}, \psi)^2 \\ &= \omega_{L/F}(-1). \end{aligned}$$

Thus we obtain

$$\gamma(1, \omega_{L/F}, \psi)^{-1} I(0) = \omega_{L/F}(-1)\chi(-1)\epsilon(\chi, \psi_0) = \epsilon(\chi, \psi_0).$$

This completes the proof. □

*Remark.* When the characteristic of $F$ is equal to 2, we can prove a similar result for a separable quadratic extension $L$ of $F$. We take as $x_0$ any element of $L$ which generates $L$ over $F$ and denote by $X^2 - rX + n$ the irreducible polynomial of $x_0$ over $F$. Then we can consider $L^\times$ as a subgroup of $\mathrm{GL}_2(F)$ by

$$a + bx_0 \longmapsto \begin{pmatrix} a & -bn \\ b & a+br \end{pmatrix}.$$

Then $\mathrm{GL}_2(F) = BL^\times$; more explicitly,

$$\begin{pmatrix} a & b \\ c & c \end{pmatrix} = \frac{1}{d^2 - rcd + nc^2} \begin{pmatrix} ad - bc & bd - rbc + nac \\ 0 & d^2 - rcd + nc^2 \end{pmatrix} \begin{pmatrix} d - cr & -cn \\ c & d \end{pmatrix}.$$

The last matrix is the image of $d - cr + cx_0$. Since $B \cap L^\times = Z$, for a character $\chi$ of $L^\times$ with $\chi|_{F^\times} = \omega_{L/F}$ we can define $f_\chi \in Ps(1, \omega_{L/F})$ in the same way as above. Let $T$ be the operator of $Ps(1, \omega_{L/F})$ to itself of order 2 defined above, and let $T f_\chi = c_\chi f_\chi$. Then we can prove

$$c_\chi = \omega_{L/F}(s)\epsilon(\chi, \psi).$$

Here, $\psi(x) = \psi_0(\mathrm{tr}(x))$ for $x \in L$. The proof proceeds in a similar way. We see that $c_\chi$ is the value at $s = 0$ of

$$\begin{aligned} &\gamma(1, \omega_{L/F}, \psi_0)^{-1} \int_F |a^2 - ra + n|^{-1/2-s} \chi(a - r + x_0)\, da \\ &= \gamma(1, \omega_{L/F}, \psi_0)^{-1} \int_F |a^2 - ra + n|^{-1/2-s} \chi(a + x_0)\, da \end{aligned}$$

and can modify the integral to obtain our result.

*Acknowledgments.* Dipendra Prasad thanks T. N. Venkataramana for discussions on the Burger-Sarnak principle and U. K. Anandavardhanan for some helpful discussions. He also thanks Hiroshi Saito for writing the appendix to this article supplying a local proof of Lemma 4.

## References

[B] D. BUMP, *Automorphic Forms and Representations*, Cambridge Stud. Adv. Math. **55**, Cambridge Univ. Press, Cambridge, 1997. MR 1431508 256

[BS] M. BURGER and P. SARNAK, *Ramanujan duals, II*, Invent. Math. **106** (1991), 1–11. MR 1123369 233, 237

[GJR] D. GINZBURG, D. JIANG, and S. RALLIS, “On the nonvanishing of the central value of the Rankin-Selberg $L$-functions, II” in *Automorphic Representations, L-Functions and Applications: Progress and Prospects*, Ohio State Univ. Math. Res. Inst. Publ. **11**, de Gruyter, Berlin, 2005, 157–191. MR 2192823 235, 253

[GP1] B. H. GROSS and D. PRASAD, *On the decomposition of a representation of* $\mathrm{SO}_n$ *when restricted to* $\mathrm{SO}_{n-1}$, Canad. J. Math. **44** (1992), 974–1002. MR 1186476 235, 251, 252, 253, 254, 255

[GP2] ———, *On irreducible representations of* $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$, Canad. J. Math. **46** (1994), 930–950. MR 1295124 253, 255

[HK] M. HARRIS and S. S. KUDLA, "On a conjecture of Jacquet" in *Contributions to Automorphic Forms, Geometry, and Number Theory*, Johns Hopkins Univ. Press, Baltimore, 2004, 355 – 371. MR 2058614 233, 234, 235, 248

[HKS] M. HARRIS, S. S. KUDLA, and W. J. SWEET, *Theta dichotomy for unitary groups*, J. Amer. Math. Soc. **9** (1996), 941 – 1004. MR 1327161 255, 258

[HL] M. HARRIS and J.-S. LI, *A Lefschetz property for subvarieties of Shimura varieties*, J. Algebraic Geom. **7** (1998), 77 – 122. MR 1620690 237

[H] H. HIJIKATA, "Any irreducible smooth $\mathrm{GL}_2$-module is multiplicity free for any anisotropic torus" in *Automorphic Forms and Geometry of Arithmetic Varieties*, Adv. Stud. Pure Math. **15**, Academic Press, Boston, 1989, 281 – 286. MR 1040610 256

[JL] H. JACQUET and R. P. LANGLANDS, *Automorphic Forms on* GL(2), Lecture Notes in Math. **114**, Springer, Berlin, 1970. MR 0401654 241

[JS1] D. JIANG and D. SOUDRY, *The local converse theorem for* $\mathrm{SO}(2n+1)$ *and applications*, Ann. of Math. (2) **157** (2003), 743 – 806. MR 1983781 251, 252

[JS2] ———, "Generic representations and local Langlands reciprocity law for $p$-adic $\mathrm{SO}_{2n+1}$" in *Contributions to Automorphic Forms, Geometry, and Number Theory*, Johns Hopkins Univ. Press, Baltimore, 2004, 457 – 519. MR 2058617 251, 252

[K] P. C. KUTZKO, *On the supercuspidal representations of* $\mathrm{GL}_2$, GL(2), Amer. J. Math. **100** (1978), 43 – 60. MR 0507253 239

[LL] J.-P. LABESSE and R. P. LANGLANDS, *L-indistinguishability for* SL(2), Canad. J. Math. **31** (1979), 726 – 785. MR 0540902 247

[L] R. P. LANGLANDS, *Base Change for* GL(2), Ann. of Math. Stud. **96**, Princeton Univ. Press, Princeton, 1980. MR 0574808 246, 255

[M] C. MOEN, *The dual pair* (U(1), U(1)) *over a p-adic field*, Pacific J. Math. **158** (1993), 365 – 386. MR 1206444 247

[P1] D. PRASAD, *Trilinear forms for representations of* GL(2) *and local $\epsilon$-factors*, Compositio Math. **75** (1990), 1 – 46. MR 1059954 233, 234, 235, 237, 239, 247

[P2] ———, *Invariant forms for representations of* GL(2) *over a local field*, Amer. J. Math. **114** (1992), 1317 – 1363. MR 1198305 254

[P3] ———, *On an extension of a theorem of Tunnell*, Compositio Math. **94** (1994), 19 – 28. MR 1302309 233, 242, 245, 246

[P4] ———, *Some applications of seesaw duality to branching laws*, Math. Ann. **304** (1996), 1 – 20. MR 1367880 247

[R] D. RAMAKRISHNAN, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* SL(2), Ann. of Math. (2) **152** (2000), 45 – 111. MR 1792292 237

[Ro] J. D. ROGAWSKI, "The multiplicity formula for $A$-packets" in *The Zeta Functions of Picard Modular Surfaces*, Univ. Montréal, Montréal, 1992, 395 – 419. MR 1155235 247

[S] H. SAITO, *On Tunnell's formula for characters of* GL(2), Compositio Math. **85** (1993), 99 – 108. MR 1199206 234, 240, 241

[Sh] F. SHAHIDI, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math. (2) **127** (1988), 547 – 584. MR 0942520 233

[T] J. B. TUNNELL, *Local $\epsilon$-factors and characters of* GL(2), Amer. J. Math. **105** (1983), 1277 – 1307. MR 0721997 234, 240, 241

[W] J.-L. WALDSPURGER, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), 173 – 242. MR 0783511 233, 240

*Prasad*
School of Mathematics, Tata Institute of Fundamental Research, Colaba, Mumbai 400005, India; dprasad@math.tifr.res.in

*Saito*
Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan; saito@math.kyoto-u.ac.jp