

# NOTES ON REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE

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These are the notes of some lectures given by the author for a workshop held at TIFR in December, 2011, giving an exposition of the Deligne-Lusztig theory. The aim of these lectures was to give an overview of the subject with several examples without burdening them with detailed proofs of the theorems involved for which we refer to the original paper of Deligne and Lusztig, as well as the beautiful book of Digne and Michel on the subject. The author thanks Shripad M. Garge for writing and texing the first draft of these notes.

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The Deligne-Lusztig theory constructs certain (virtual) representations of  $G(\mathbb{F}_q)$  denoted as  $R_G(T, \theta)$  where  $G$  is a connected reductive algebraic group defined over  $\mathbb{F}_q$ ,  $T$  is a maximal torus in  $G$  defined over  $\mathbb{F}_q$  and  $\theta$  is a character of  $T(\mathbb{F}_q)$ ,  $\theta : T(\mathbb{F}_q) \rightarrow \mathbb{C}^\times \xrightarrow{\sim} \overline{\mathbb{Q}}_l^\times$ . The representation  $R_G(T, \theta)$  is called the Deligne-Lusztig induction of the character  $\theta$  of  $T(\mathbb{F}_q)$  to  $G(\mathbb{F}_q)$ , generalizing parabolic induction with which it coincides when  $T$  is a maximally split maximal torus in  $G$ . It is known that if  $T \subseteq M$  where  $M$  is a Levi subgroup of a parabolic subgroup  $P$  of  $G$  defined over  $\mathbb{F}_q$ , then  $R_G(T, \theta) = \text{Ind}_P^G(R_M(T, \theta))$ ; thus  $R_G(T, \theta)$  are really new objects only for those maximal tori which are not contained inside any Levi subgroup of a parabolic subgroup  $P$  of  $G$  defined over  $\mathbb{F}_q$ . These are exactly those tori which do not contain a non-central split torus of  $\dim \geq 1$ .

An irreducible representation  $\pi$  of  $G$  is said to be *cuspidal* if it does not appear as a factor of  $\text{Ind}_P^G \rho$  where  $P$  is a proper parabolic and  $\rho$  is an irreducible representation of  $M$  considered as a representation of  $P$ . Any representation of  $G(\mathbb{F}_q)$  can be understood in terms of cuspidal representations of  $G(\mathbb{F}_q)$ , and of Levi subgroups of proper parabolic subgroups defined over  $\mathbb{F}_q$ . An important part of the subject is to study the *principal series representations*,  $\text{Ind}_P^G \rho$ , where  $\rho$  is an irreducible representation of  $M$ . The decomposition of a principal series representation into irreducible components is in general a difficult problem if one is interested in complete character information, but at least their parametrization can be understood.

## 1. PRELIMINARIES ON ALGEBRAIC GROUPS

Since Deligne-Lusztig theory constructs representations of  $G(\mathbb{F}_q)$  using characters of all maximal tori in  $G$  defined over  $\mathbb{F}_q$ , we begin with a review of algebraic groups and their maximal tori.

An algebraic group over a field  $k$ , denoted by  $G/k$ , gives rise among other things to  $G(A)$  which are groups for any algebra  $A/k$  associated in a functorial way, i.e., for every morphism of algebras  $A_1 \rightarrow A_2$ , we get a morphism of groups  $G(A_1) \rightarrow G(A_2)$ . An algebraic group  $G$  is said to be *reductive* if  $G \subseteq \mathrm{GL}(V)$ ,  $V$  a finite dimensional vector space over  $k$ , and the representation  $V$  of  $G$  is completely reducible. (In positive characteristic, we are not suggesting that *any* representation is completely reducible, but that there is one faithful representation which is completely reducible.)

We say that  $G$  is *semisimple* if  $G$  is connected and there is no nontrivial homomorphism of algebraic groups  $G \rightarrow A$  with  $A$  abelian.

A good example of a reductive group to keep in mind –although too simple for the purposes of these notes– is  $G = \mathrm{GL}_n(\mathbb{F}_q)$  and  $T = \prod_{i=1}^d K_i^\times$  where  $K_i$  are extensions of  $\mathbb{F}_q$  of degree  $n_i$  such that  $\sum n_i = n$ . Among the maximal tori, there is the split torus,

$$T = (\mathbb{F}_q^\times)^n = \begin{pmatrix} * & & & \\ & * & & \\ & & \ddots & \\ & & & * \end{pmatrix} \subseteq \mathrm{GL}_n(\mathbb{F}_q)$$

and one anisotropic torus (mod center),  $\mathbb{F}_{q^n}^\times \subseteq \mathrm{GL}_n(\mathbb{F}_q)$ . A Borel subgroup of  $\mathrm{GL}_n(\mathbb{F}_q)$  is up to conjugacy the subgroup  $B$  of the group of upper triangular matrices:

$$B = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & \ddots & \vdots \\ & & & * \end{pmatrix},$$

and parabolic subgroups are (up to conjugacy) those subgroups of  $G$  that contain this Borel subgroup. It can be proved that subgroups of  $G$  containing the group of upper triangular matrices are subgroups of the form

$$P_\alpha = \begin{pmatrix} \mathrm{GL}_{d_1} & * & * & * \\ & \mathrm{GL}_{d_2} & * & * \\ & & \ddots & \vdots \\ & & & \mathrm{GL}_{d_k} \end{pmatrix}$$

where  $\alpha = (d_1, \dots, d_k)$  with  $d_i \geq 1$ ,  $\sum d_i = n$ . We have a decomposition, called the *Levi decomposition*,  $P_\alpha = M_\alpha N_\alpha$  where  $M_\alpha = \prod \mathrm{GL}_{d_i}(\mathbb{F}_q)$  is called a *Levi subgroup* of  $P_\alpha$  and  $N_\alpha$  is the unipotent radical of  $P_\alpha$ . In fact, one has a split exact sequence of groups  $1 \rightarrow N_\alpha \rightarrow P_\alpha \rightarrow M_\alpha \rightarrow 1$ .

Classification of reductive algebraic groups over  $k$  is usually done in two steps:

1. The theory for  $k = \bar{k}$ : Here there are no surprises and the theory is the same as for compact, connected, Lie groups, or for  $k = \mathbb{C}$ . Very briefly put, the theory says that any connected, reductive group over  $k = \bar{k}$  is a product of groups of the type  $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$  together with some center. We have the classical groups

$$A_n \longleftrightarrow \mathrm{SL}_{n+1}, \quad B_n \longleftrightarrow \mathrm{SO}_{2n+1}, \quad C_n \longleftrightarrow \mathrm{Sp}_{2n}, \quad D_n \longleftrightarrow \mathrm{SO}_{2n}$$

and the remaining groups,  $G_2, F_4, E_6, E_7, E_8$ , are called exceptional groups.

2. Rationality questions: The next step in the classification program is to understand what are  $G'/k$  for a given  $G/k$  such that  $G/\bar{k} = G'/\bar{k}$ .

This question is intimately related to the arithmetic of the field  $k$ . This is studied among other cases in the following special cases:

- i) for  $k = \mathbb{R}$ : This was completed by E. Cartan more than a century back. For example, forms of  $\mathrm{SO}(n)$  are  $\mathrm{SO}(p, q)$  (Sylvester's law of inertia).
- ii) for  $k = \mathbb{Q}_p$ : A complete understanding is known.
- iii) for  $k = \mathbb{F}_q$ : Here there are in most cases the only split groups, and in the other cases, one more which is quasi-split. The groups of type  $B_n = \mathrm{SO}_{2n+1}(\mathbb{F}_q)$ ,  $C_n = \mathrm{Sp}_{2n}(\mathbb{F}_q)$ ,  $G_2$ ,  $F_4$ ,  $E_7$  and  $E_8$  have only the split form over  $\mathbb{F}_q$  whereas for the groups of type  $A_n$ ,  $D_n$  and  $E_6$  we have quasi-split form also:

$$A_n \longleftrightarrow \mathrm{SL}_{n+1}(\mathbb{F}_q), \mathrm{SU}_{n+1}(\mathbb{F}_q), \quad D_n \longleftrightarrow \mathrm{SO}_{2n}^{\pm}(\mathbb{F}_q),$$

where  $+$  denotes the isometry group of a split quadratic form and  $-$  denotes the quasi-split but non-split group which is the isometry group of the norm form of a quadratic extension added by  $(2n - 2)$ -dimensional split quadratic space, while for  $\mathrm{SO}_8$  we have one more form, because the Dynkin diagram of  $\mathrm{SO}_8$  has an extra automorphism.

An important invariant associated to  $G/\mathbb{F}_q$  is its order,  $|G(\mathbb{F}_q)|$ , e.g.  $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^n (q^i - 1)$ . The Weyl group of  $G$  is  $W(G) := N(T)(\mathbb{F}_q)/T(\mathbb{F}_q)$  for a maximally split maximal torus  $T$  in  $G$ . The Weyl groups for classical groups are given below. The Weyl group of  $G_2$  is the dihedral group  $D_6$  (of order 12) and the Weyl groups of other exceptional groups are slightly more difficult to describe.

$$W(A_n) = S_{n+1}, \quad W(B_n) = W(C_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \quad \text{and} \quad W(D_n) = (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n.$$

The Weyl group operates on  $T$ , and hence on the character group  $X^*(T) = \mathrm{Hom}(T, G_m) \cong \mathbb{Z}^l$ . Therefore  $W$  operates on the polynomial algebra  $S^*(X^*(T)_{\mathbb{R}})$  on  $X^*(T)_{\mathbb{R}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ . It is a theorem that  $S^*(X^*(T)_{\mathbb{R}})^W = \langle p_1, \dots, p_l \rangle$  where  $p_i$  are homogeneous polynomials of degree  $d_i$  called exponents of  $G$ . These  $d_i$  are:

$$\begin{aligned} A_n &: 2, 3, \dots, (n+1). \\ B_n, C_n &: 2, 4, 6, \dots, 2n. \\ D_n &: 2, 4, \dots, (2n-2), n. \\ G_2 &: 2, 6. \\ F_4 &: 2, 6, 8, 12. \\ E_6 &: 2, 5, 6, 8, 9, 12. \\ E_7 &: 2, 6, 8, 10, 12, 14, 18. \\ E_8 &: 2, 8, 12, 14, 18, 20, 24, 30. \end{aligned}$$

A basic result about these exponents is that  $\prod_{i=1}^l d_i = |W|$  and  $\sum_{i=1}^l (d_i - 1) = N$  is the number of positive roots.

**Theorem 1.1.** (1) If  $G$  is split then  $|G(\mathbb{F}_q)| = q^N \prod_{i=1}^l (q^{d_i} - 1)$ .

$$(2) \lim_{q \rightarrow 1} \frac{|G(\mathbb{F}_q)|}{(q-1)^l} = \prod_{i=1}^l d_i = |W|.$$

**Proposition 1.1.**  $|\mathrm{U}_n(\mathbb{F}_q)| = |\mathrm{GL}_n(\mathbb{F}_{-q})|$ .

**Remark 1.2.** One can prove this proposition by elementary methods by directly calculating the order of the two groups involved. A proof via  $l$ -adic cohomology of the flag variety  $G/B$  would be desirable which would then generalize to other quasi-split but non-split groups. (The recipe relating the orders of  $\mathrm{GL}_n(\mathbb{F}_q)$  and  $\mathrm{U}_n(\mathbb{F}_q)$  works also for outer form of  $E_6(\mathbb{F}_q)$ , and also for  $D_n$ , but only for  $n$  odd.) It is part of what's called *Ennola duality* that one can go from the character table of  $\mathrm{GL}_n(\mathbb{F}_q)$  to  $\mathrm{U}_n(\mathbb{F}_q)$  by changing  $q$  to  $-q$ .

## 2. CLASSICAL GROUPS AND TORI

Suppose  $A$  is a semisimple algebra over a field  $k$ , i.e., an algebra which is isomorphic to a finite direct product of  $\mathbb{M}_n(D)$  where  $D$ 's are division algebras over  $k$ . We note that a semisimple algebra is canonically

a product of simple algebras, in particular, a commutative semisimple algebra is canonically a product of field extensions of  $k$ . Suppose  $x \mapsto x^*$  is an involution on  $A$ , i.e., we have

$$(x + y)^* = x^* + y^*, \quad (xy)^* = y^*x^* \quad \text{and} \quad (x^*)^* = x.$$

The involution  $*$  preserves  $k$  and may act as identity on  $k$ , in which case it is said to be of first type, or may act nontrivially on  $k$ , in which case it is said to be of second type. For an involution  $*$  on a semisimple algebra  $A$ , we define  $G(A, *) = \{x \in A : xx^* = 1\}$ .

A classical group is a group of the form  $G(A, *)$ . It is a theorem that any form of  $A_n, B_n, C_n$  or  $D_n$ , except the triality forms of  $D_4$  are groups of the form  $G(A, *)$ .

Let  $C$  be a commutative semisimple subalgebra of  $A$  of maximal dimension invariant under  $*$ .

**Lemma 2.1.** *The group  $T(C) := \{x \in C : xx^* = 1\}$  is a maximal torus in  $G(A, *)$ , and any maximal torus of  $G(A, *)$  is of this form.*

*Proof.* We begin by noting that given a maximal torus  $T$  in  $G(A, *)$ , the subalgebra of  $A$  generated by a generic element of  $T$ , if  $k$  is infinite, or in general the centralizer of  $T$  in  $A$ , is a maximal commutative subalgebra of  $A$  of the correct dimension (as one can prove assuming  $k$  algebraically closed). The converse follows by an explicit calculation assuming  $k$  to be algebraically closed.  $\square$

**Lemma 2.2.** *Let  $(C, *)$  be any commutative semisimple algebra with involution over a field  $k$  of dimension  $n$  such that in the decomposition of  $C$  as a direct sum of field extensions of  $k$ , there is at most one factor on which the involution acts trivially, and that factor is  $k$  (this forces  $n$  to be odd). Fixing a unit  $d$  in  $C$  with  $d^* = \epsilon d$ ,  $\epsilon = \pm 1$ , we can construct a  $\epsilon$ -hermitian form on  $C$  by  $\langle c_1, c_2 \rangle_d = \text{tr}_{C/k}(c_1 d c_2^*)$  with  $T(C)$  a maximal torus of the corresponding unitary group.*

*Proof.* Note that any commutative algebra  $C$  with involution can be embedded in the semisimple algebra  $A$ , with  $\dim C = \sqrt{\dim A}$ , consisting of the algebra of  $k$  endomorphisms of  $C$  treated as a vector space over  $k$ . We use the involution on  $C$  together with the element  $d \in C$  with  $d^* = \epsilon d$ ,  $\epsilon = \pm 1$ , to construct an involution to be denoted  $* = *_d$  on  $A$ . The vector space  $C$  comes equipped with a bilinear form given by  $\langle c_1, c_2 \rangle_d = \text{tr}_{C/k}(c_1 \cdot d \cdot c_2^*)$ . This bilinear form is  $\epsilon$ -hermitian, i.e.,

$$\langle c_1, c_2 \rangle_d = \text{tr}_{C/k}(c_1 \cdot d \cdot c_2^*) = \epsilon [\text{tr}_{C/k}(c_2 \cdot d \cdot c_1^*)]^* = \epsilon \langle c_2, c_1 \rangle_d^*.$$

If the involution on  $C$  is of the first kind, fixing a nonzero element  $d$  in  $C$  with  $d^* = -d$ , we can construct a symplectic form on  $C$  by  $\langle c_1, c_2 \rangle_d = \text{tr}_{C/k}(c_1 d c_2^*)$ .

The bilinear form  $\langle c_1, c_2 \rangle_d$  on  $C$  induces a natural involution  $a \rightarrow a^*$  on  $A$  with the property that  $\langle a c_1, c_2 \rangle_d = \langle c_1, a^* c_2 \rangle_d$ . Under the right regular representation of  $C$  on itself,  $C$  embeds inside  $A$  as a maximal commutative subalgebra, invariant under the involution on  $A$  defined here since

$$\langle c \cdot c_1, c_2 \rangle_d = \text{tr}_{C/k}(c c_1 d c_2^*) = \text{tr}_{C/k}(c_1 d [c^* c_2]^*) = \langle c_1, c^* c_2 \rangle_d.$$

It is easy to check that  $T(C) := \{x \in C : xx^* = 1\}$  is a maximal torus in  $G(A, *)$ , and the converse follows from the previous Lemma.  $\square$

**Remark :** For groups  $G = G(A, *)$  which have a unique  $k$ -rational form, in which case  $A$  is unique too, such as for symplectic groups over any field, or orthogonal groups in odd dimension, symplectic groups, and unitary groups over finite fields, this allows one to identify all maximal tori of these classical groups. For embedding  $T(C)$  in a *given* orthogonal group in even number of variables over a finite field, there is an obstruction in terms of the discriminant of  $(A, *)$ . For general fields  $k$ , we are not able to say which hermitian or quadratic forms arise as  $\langle c_1, c_2 \rangle_d = \text{tr}_{C/k}(c_1 \cdot d \cdot c_2^*)$  as  $d$  varies among the units of  $C$  with  $d^* = \pm d$ , which are presumably all the hermitian or quadratic forms for which  $T(C)$  is a maximal torus in the corresponding unitary group.

**Remark :**

- (1) Any maximal torus in  $\text{GL}_n(k)$  is of the form  $\prod K_i^\times$  where  $K_i/k$  are field extensions of  $k$  with  $\sum [K_i : k] = n$ .

- (2) Any involution on  $C = \prod K_i$  when acting on a particular  $K_i$  takes it to a  $K_j$  or preserves it, so we have  $C = \prod_I (K_i \times K_i) \times \prod_J K_j$ . For an extension  $K/k$  which carries an involution  $\iota$  with fixed field  $K^\iota$ , we denote by  $K^1$  the kernel of the norm map  $K^\times \rightarrow K^{\iota \times}$ . Then any maximal torus in  $G(A, *)$  is of the form  $T(C) = \prod_I K_i^\times \times \prod_J K_j^1$ .
- (3) For unitary groups one gets a commutative subalgebra  $C$  left invariant under an involution of the second kind. Define  $C^\circ = \{x \in C : x^* = x\}$ . Then  $k \otimes C^\circ \cong C$  as  $*$ -algebras. This gives a bijection between tori in  $\mathrm{GL}_n(k)$  (defined by  $C^\circ$ ) and those in  $\mathrm{U}_n(k)$  (defined by  $C$ ) for  $k$  a finite field.

### 3. SOME EXCEPTIONAL GROUPS

We saw above that a classical group can be described in terms of a semisimple, associative algebra. For exceptional groups, we need to consider non-associative algebras. This section plays no role in the rest of these notes, and are included just give some flavor of the groups involved in these cases.

Construction of exceptional groups begins with an *octonian* algebra  $\mathbb{O}$  over  $k$  which we first define. These are non-commutative, non-associative algebras but are endowed with an involution (i.e., an anti-automorphism)  $x \mapsto \bar{x}$ , such that  $\mathrm{tr}(x) := x + \bar{x} \in k$ ,  $\mathrm{Nm}(x) := x\bar{x} \in k$ , which is a nondegenerate quadratic form with  $\mathrm{Nm}(xy) = \mathrm{Nm}(x)\mathrm{Nm}(y)$ . One can realize the split octonian algebra  $\mathbb{O}$  in the matrix form as follows:

$$\mathbb{O} = \begin{pmatrix} k & V \\ V^\vee & k \end{pmatrix},$$

where  $V$  is a 3-dimensional vector space over  $k$  with dual space  $V^\vee$ , together with a fixed isomorphism of  $\Lambda^3 V$  with  $k$ , and hence also of  $\Lambda^3 V^\vee$  with  $k$ , with multiplication defined by,

$$\begin{pmatrix} a & v \\ w & b \end{pmatrix} \begin{pmatrix} a' & v' \\ w' & b' \end{pmatrix} = \begin{pmatrix} aa' + \langle v, w' \rangle & av' + b'v - w \times w' \\ a'w + bw' + v \times v' & bb' + \langle v', w \rangle \end{pmatrix},$$

where the products  $\langle v, w' \rangle$  denotes the usual pairing between  $V$  and  $V^\vee$  and for vectors  $v', v'' \in V$ ,  $v' \times v''$  is the vector in  $V^\vee$  with the property that  $\langle v, v' \times v'' \rangle = v \wedge v' \wedge v'' \in k$  (through the fixed isomorphism of  $\Lambda^3 V$  with  $k$ ); similarly, for vectors  $w', w'' \in V^\vee$ ,  $w' \times w'' \in V$ .

There are other, non-split, octonian algebras too defined in terms of 3-dimensional Hermitian vector spaces, but we will not get into that here.

The automorphism group of an octonion algebra is a group of type  $G_2$ . Thus  $G_2$  comes equipped with a natural 7 dimensional representation over  $k$  on trace zero elements of  $\mathbb{O}$ .

The descriptions of other exceptional groups involve exceptional Jordan algebras,  $J = J(\mathbb{O})$ . It is a 27 dimensional non-commutative, non-associative algebra over  $k$ . An element of  $J$  is of the following form:

$$A = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix}$$

where  $a, b, c \in k$  and  $x, y, z \in \mathbb{O}$ . We define a new product on  $J$ ,  $\alpha \circ \beta = \frac{1}{2}(\alpha\beta + \beta\alpha)$ . Then with this multiplication on  $J$ , the automorphism group of the algebra  $J$  is the group  $F_4$ . Thus  $F_4$  comes equipped with a natural 26 dimensional representation over  $k$  on trace zero elements of  $J$ .

We have a map  $\det : J \rightarrow k$ , obtained by taking the usual determinant of an element, which lies in  $\mathbb{O}$ , and then taking its trace from  $\mathbb{O}$  to  $k$ , given by:

$$\det A = abc + \mathrm{tr}(xyz) - a\mathrm{Nm}(x) - b\mathrm{Nm}(y) - c\mathrm{Nm}(z).$$

The group  $\mathrm{Aut}(J, \det)$  is the group  $E_6$  which comes equipped with a natural 27 dimensional representation over  $k$ .

## 4. BASIC NOTIONS IN REPRESENTATION THEORY

Let  $G$  be a reductive group,  $P = MN$  be a proper parabolic in  $G$ . Associated to a representation  $\pi$  of  $G(\mathbb{F}_q)$  there is a representation, called the *Jacquet module*  $\pi_N$ , of  $M(\mathbb{F}_q)$  which is  $\pi^{N(\mathbb{F}_q)}$ , the subspace of  $N(\mathbb{F}_q)$ -fixed vectors in  $\pi$ . Since  $N(\mathbb{F}_q)$  is normalized by  $M(\mathbb{F}_q)$ ,  $\pi_N$  is a representation of  $M(\mathbb{F}_q)$ . Jacquet modules play a very large role in representation theory.

A representation  $\pi$  of  $G(\mathbb{F}_q)$  is called *cuspidal* if  $\pi_N = 0$  for unipotent radicals  $N$  of every proper parabolic.

According to Harish-Chandra's philosophy of cusp forms, cuspidal representations are building blocks of all representations as the following theorem shows:

**Theorem 4.1.** *Let  $\pi$  be an irreducible representation of  $G(\mathbb{F}_q)$ . Then there exists a unique associated class of parabolics  $P = MN$  and a unique cuspidal representation  $\rho$  of  $M(\mathbb{F}_q)$  such that*

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(\pi, \mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho) \neq 0.$$

Here uniqueness means that the pair  $(M, \rho)$  is unique up to conjugacy by  $G(\mathbb{F}_q)$ . Because of this theorem, the classification of  $G(\mathbb{F}_q)$  splits into two parts:

- (1) Classify cuspidal representations of all Levi subgroups including  $G$ ;
- (2) Understand the decomposition of  $\mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho$  for  $\rho$  a cuspidal representation of  $M(\mathbb{F}_q)$ .

Green answers both these parts in his paper for  $\mathrm{GL}_n(\mathbb{F}_q)$ .

For (1), cuspidal representations of  $\mathrm{GL}_n(\mathbb{F}_q)$  are in one-to-one correspondence with regular characters  $\chi : \mathbb{F}_{q^n}^\times \rightarrow \mathbb{C}^\times$ , up to Galois action on  $\mathbb{F}_{q^n}^\times$ . A character  $\chi$  of  $\mathbb{F}_{q^n}^\times$  is called *regular* if  $\chi^\sigma \neq \chi$  for all  $1 \neq \sigma \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ .

**Theorem 4.2.** *For a regular character  $\chi : \mathbb{F}_{q^n}^\times \rightarrow \mathbb{C}^\times$  there exists an irreducible cuspidal representation  $\pi_\chi$  of  $\mathrm{GL}_n(\mathbb{F}_q)$ , such that for its character  $\Theta_{\pi_\chi}$ , we have:*

- i)  $\Theta_{\pi_\chi}(su) = 0$  if  $s \notin \mathbb{F}_{q^n}^\times \hookrightarrow \mathrm{GL}_n(\mathbb{F}_q)$ .
- ii) If  $s$  generates a subfield  $\mathbb{F}_{q^d} \subseteq \mathbb{F}_{q^n}$

$$\Theta_{\pi_\chi}(su) = (-1)^n (1 - q^d)(1 - q^{2d}) \cdots (1 - q^{td}) \sum_{\sigma \in \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)} \chi(s^\sigma)$$

where  $t + 1 = \dim \ker(u - 1)$  inside  $\mathrm{GL}_m(\mathbb{F}_{q^d})$ , the dimension being counted over  $\mathbb{F}_{q^d}$ ,  $n = md$ .

To understand the decomposition of  $\mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho$  for  $\rho$  a cuspidal representation of  $M(\mathbb{F}_q)$ , and  $G = \mathrm{GL}_n(\mathbb{F}_q)$ , it suffices<sup>1</sup> to understand the decomposition of  $\sigma \times \cdots \times \sigma$ , a representation of  $\mathrm{GL}_{dm}(\mathbb{F}_q)$  obtained by parabolic induction of the representation  $\sigma \otimes \cdots \otimes \sigma$  of  $\mathrm{GL}_m(\mathbb{F}_q)^d$  considered as a Levi subgroup of  $\mathrm{GL}_{dm}(\mathbb{F}_q)$ . The problem gets reduced to understanding the case when  $\sigma = 1, m = 1$ . At least there exists a bijective correspondence between irreducible complex representations of  $\sigma \times \cdots \times \sigma$  and  $1 \times \cdots \times 1$ . Finally  $1 \times \cdots \times 1$  is nothing but  $\mathrm{Ind}_B^G 1$ .

**Theorem 4.3.**  $\mathrm{End}(\mathrm{Ind}_B^G 1, \mathrm{Ind}_B^G 1) \cong \mathbb{C}[W]$ .

**Corollary 4.4.** *Irreducible components of  $\mathrm{Ind}_B^G 1$  are in one-to-one correspondence with  $\widehat{W}$ , the set of irreducible representations of  $W$ .*

Unipotent conjugacy classes in  $\mathrm{GL}_n(\mathbb{F}_q)$  are in one-to-one correspondence with partitions of  $n$  which also parametrize conjugacy classes in  $S_n$  which is the Weyl group of  $\mathrm{GL}_n(\mathbb{F}_q)$ .

<sup>1</sup>One observes that if we have distinct cuspidal representations  $\rho_i$ , then the principal series representation  $\rho_1 \times \cdots \times \rho_d$  is irreducible.

## 5. THE DELIGNE-LUSZTIG THEORY

Let us fix some notation. For any integer  $n$ , we denote by  $n_p$ , the  $p$ -primary component of  $n$ , and  $n_{p'} = n/n_p$ . Define a sign  $\epsilon_G$  associated to a reductive group  $G$  which plays a large role in character theory by  $\epsilon_G := (-1)^{r(G)}$ , where  $r(G)$  is the dimension of the maximal  $\mathbb{F}_q$ -split torus in  $G$ , the  $\mathbb{F}_q$ -rank of  $G$ .

The work of Deligne and Lusztig was motivated by the following conjecture due to Macdonald.

**Conjecture 5.1 (Macdonald).** *To every maximal torus  $T/\mathbb{F}_q$  contained in  $G/\mathbb{F}_q$  and a character  $\theta : T(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ , which is regular in the sense that  $w\theta \neq \theta$  for all  $w \in W := N(T)(\mathbb{F}_q)/T(\mathbb{F}_q)$ ,  $w \neq 1$ , there exists an irreducible representation  $\pi_\theta$  of  $G(\mathbb{F}_q)$  of dimension  $|G(\mathbb{F}_q)|_{p'}/|T(\mathbb{F}_q)|$  whose value on a regular semisimple element  $s \in G(\mathbb{F}_q)$  is nonzero only if  $s \in T(\mathbb{F}_q)$  up to conjugacy. The character  $\chi_\theta$  of  $\pi_\theta$  at  $s \in T(\mathbb{F}_q)$  is  $\epsilon_T \epsilon_G \sum_{w \in W} \theta(s^w)$ . Further, the character  $\chi_\theta$  at unipotent elements is independent of  $\theta$ .*

A part of the conjecture was that ‘most’ of the representations of  $G(\mathbb{F}_q)$  are irreducible  $\pi_\theta$ .

The work of Deligne-Lusztig proves Macdonald’s conjecture and gives a geometric way of realizing them.

**Example 5.2.** (1) The dimension of a cuspidal representation of  $\mathrm{GL}_n(\mathbb{F}_q)$  is  $\prod_{i=1}^{n-1} (q^i - 1) = | \mathrm{GL}_n(\mathbb{F}_q) |_{p'}/|T(\mathbb{F}_q)|$  where  $T(\mathbb{F}_q) = \mathbb{F}_q^\times$ .  
(2) For  $\mathrm{GL}_2(\mathbb{F}_q)$ :  $Ps(\alpha, \beta)(x, y) = (\alpha(x)\beta(y) + \alpha(y)\beta(x))$ , where  $\alpha, \beta$  are characters of  $\mathbb{F}_q^\times$ .  
(3) For  $\mathrm{GL}_2(\mathbb{F}_q)$ :  $Ds(\chi)(x) = -(\chi(x) + \chi(x^\sigma))$ , where  $\chi$  is a regular character of  $\mathbb{F}_q^\times$ .

Instead of defining  $R_T^G(\theta)$  for any character  $\theta$  of  $T(\mathbb{F}_q)$ , we define a more general representation  $R_L^G(\rho)$  where  $\rho$  is an irreducible representation of  $L(\mathbb{F}_q)$ , where  $L$  is a subgroup of  $G$  defined over  $\mathbb{F}_q$ , which is a Levi subgroup of a parabolic  $P$  of  $G$  defined over  $\mathbb{F}_q$ , without requiring that  $L$  is the Levi component of a parabolic defined over  $\mathbb{F}_q$ .

Begin with the algebraic variety  $X_U = \{g \in G(\overline{\mathbb{F}_q}) : g^{-1}F(g) \in U(\overline{\mathbb{F}_q})\}$  where  $P = LU$  is a parabolic in  $G(\overline{\mathbb{F}_q})$ , and  $F(g)$  denotes the image of  $g \in G(\overline{\mathbb{F}_q})$  under the Frobenius map. The variety  $X_U$  has left action of  $G(\mathbb{F}_q)$ , and right action of  $L(\mathbb{F}_q)$ . The Deligne-Lusztig representation is on the  $\overline{\mathbb{Q}_l}$ -vector space  $H_c^*(X) = \sum (-1)^i H_c^i(X_U, \overline{\mathbb{Q}_l})$  admitting an action of  $G(\mathbb{F}_q) \times L(\mathbb{F}_q)$ . For an irreducible representation  $\rho$  of  $L(\mathbb{F}_q)$ , the representation  $R_L^G(\rho)$  is the  $\rho$ -isotypic component of this representation which is a (virtual) representation of  $G(\mathbb{F}_q)$ . As the notation suggests,  $R_L^G(\rho)$  depends only on  $L$  and not on the parabolic  $P$  (defined over  $\overline{\mathbb{F}_q}$ ) which is used to define  $R_L^G(\rho)$ .

We define a pairing  $\langle -, - \rangle$  on the Grothendieck group of representations of a group by  $\langle \pi, \pi' \rangle = \delta_{\pi, \pi'}$  for irreducible representations  $\pi, \pi'$  and then extended by linearity to the Grothendieck group. One then has:

**Theorem 5.3.**  $\langle R_T^G(\theta), R_{T'}^G(\theta') \rangle = \frac{1}{|T(\mathbb{F}_q)|} \left[ \{g \in G(\mathbb{F}_q) : {}^gT = T', {}^g\theta = \theta'\} \right]$

Two pairs  $(T, \theta)$  and  $(T', \theta')$  are said to be *geometrically conjugate* if there exists  $n \in \mathbb{N}$  such that  $(T_n, \theta_n)$  and  $(T'_n, \theta'_n)$  are conjugate. Here for a maximal torus  $T$  of  $G$  defined over  $\mathbb{F}_q$  and a character  $\theta : T(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ , the torus  $T_n$  is the torus  $T$  now considered in  $G(\mathbb{F}_{q^n})$ , and the character  $\theta_n$  is the character of  $T_n(\mathbb{F}_q)$  given by  $T_n(\mathbb{F}_q) = T(\mathbb{F}_{q^n}) \xrightarrow{\mathrm{Nm}} T(\mathbb{F}_q) \xrightarrow{\theta} \mathbb{C}^\times$ , where  $\mathrm{Nm} : T(\mathbb{F}_{q^n}) \rightarrow T(\mathbb{F}_q)$  is the norm mapping.

**Theorem 5.4.** *The virtual representations  $R_T^G(\theta)$  and  $R_{T'}^G(\theta')$  are disjoint if and only if the pairs  $(T, \theta)$  and  $(T', \theta')$  are not geometrically conjugate.*

The representation  $R_L^G(\rho)$  is nothing but the parabolic induction if  $L$  is a Levi subgroup of a parabolic  $P$  which is defined over  $\mathbb{F}_q$ . In that case,  $X_U$  can be identified to  $G(\mathbb{F}_q) \times_{U(\mathbb{F}_q)} U$ . The group  $U$  being contractible, this space is for topological purposes essentially  $G(\mathbb{F}_q)/U(\mathbb{F}_q)$  which is the home for the principal series representations.

The representations  $R_L^G(\rho)$  have following nice properties:

1) Transitivity: If  $L$  is a Levi subgroup in  $G$  and  $M$  is a Levi subgroup of  $L$  then  $R_L^G(R_M^L(\rho)) = R_M^G(\rho)$ .

2) Behaviour under morphisms: If  $\pi : G_1 \rightarrow G_2$  is an isomorphism up to centers, i.e., associated mapping from  $G_1/Z_1 \rightarrow G_2/Z_2$  is an isomorphism of algebraic groups, then under such a morphism, tori in  $G_1$  and  $G_2$  correspond in a natural way (as the inverse image under  $\pi$  of a maximal torus in  $G_2$  is a maximal torus in  $G_1$ , and every maximal torus in  $G_1$  arises in this way). We then have the following:

For a torus  $T_2 \subseteq G_2$ , let  $\pi^{-1}(T_2) = T_1 \subseteq G_1$ . Thus we have a homomorphism of groups  $T_1(\mathbb{F}_q) \rightarrow T_2(\mathbb{F}_q)$ . For  $\theta_2 : T_2(\mathbb{F}_q) \rightarrow \mathbb{C}^\times$ , composing with the homomorphism of groups  $T_1(\mathbb{F}_q) \rightarrow T_2(\mathbb{F}_q)$ , we get a character of  $T_1(\mathbb{F}_q)$ , which we denote as  $\theta_2|_{T_1}$ . Then one has  $R_{T_2}^{G_2}(\theta_2)|_{G_1} = R_{T_1}^{G_1}(\theta_2|_{T_1})$ .

We have following corollaries of the two properties listed above:

**Corollary 5.5.** (1) If  $T \subseteq L$  with  $L$  a Levi subgroup of a parabolic subgroup of  $G$  defined over  $\mathbb{F}_q$ ,  $R(T, \theta) := R_T^G(\theta)$  has no cuspidal components, and therefore, the only tori  $T$  for which  $R(T, \theta)$  may have cuspidal representations are anisotropic mod center.

(2)  $D$ - $L$  representations of  $\mathrm{SL}_n(\mathbb{F}_q)$  are nothing but restrictions of  $D$ - $L$  representations of  $\mathrm{GL}_n(\mathbb{F}_q)$ .

**Proposition 5.1.** Suppose  $\pi = \pi_\chi$  is an irreducible cuspidal representation of  $\mathrm{GL}_n(\mathbb{F}_q)$  associated to  $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$ . Write the principal series representation  $Ps(\pi, \pi)$  of  $\mathrm{GL}_{2n}(\mathbb{F}_q)$  as  $St_2(\pi) + Id_2(\pi)$  with  $\dim St_2(\pi) \geq \dim Id_2(\pi)$ . Then  $\dim St_2(\pi) = q^n \dim Id_2(\pi)$ .

*Proof.* We will prove below that  $St_2(\pi) - Id_2(\pi)$  as a virtual character is the same as  $R_{\mathbb{F}_q^\times}^{\mathrm{GL}_{2n}(\mathbb{F}_q)}(\chi \circ \mathrm{Nm})$ ,

where  $\chi \circ \mathrm{Nm}$  is the character of  $\mathbb{F}_{q^{2n}}^\times$  given by:  $\mathbb{F}_{q^{2n}} \xrightarrow{\mathrm{Nm}} \mathbb{F}_{q^n} \xrightarrow{\chi} \mathbb{C}^\times$ . In particular,

$$\dim[St_2(\pi) - Id_2(\pi)] = (q^{2n-1} - 1) \cdots (q - 1).$$

Since,

$$\dim Ps(\pi, \pi) = \frac{|\mathrm{GL}_{2n}(\mathbb{F}_q)|_{p'}}{|T(\mathbb{F}_q)|} = \frac{(q^{2n} - 1) \cdots (q - 1)}{(q^n - 1)^2},$$

we find that,

$$\dim St_2(\pi) = (q - 1) \cdots (q^{2n-1} - 1) \left[ \frac{(q^{2n} - 1)}{(q^n - 1)} + 1 \right] = \frac{2q^n}{(q^n - 1)}$$

and

$$\dim Id_2(\pi) = (q - 1) \cdots (q^{2n-1} - 1) \left[ \frac{(q^{2n} - 1)}{(q^n - 1)} - 1 \right] = \frac{2}{(q^n - 1)}.$$

In particular,  $\dim St_2(\pi) / \dim Id_2(\pi) = q^n$ . □

We now prove the observation that was used here:  $St_2(\pi) - Id_2(\pi)$  as a virtual representation is the same as  $R_{\mathbb{F}_q^\times}^{\mathrm{GL}_{2n}(\mathbb{F}_q)}(\chi \circ \mathrm{Nm})$ .

*Proof.* Recall that  $R_{T_1}^G(\theta_1)$  and  $R_{T_2}^G(\theta_2)$  are orthogonal unless  $T_1 = T_2$ ,  $\theta_1 = \theta_2$ , up to conjugacy by  $G(\mathbb{F}_q)$ , and disjoint if and only if geometrically distinct. It follows that the representations  $R_{\mathbb{F}_q^\times \times \mathbb{F}_q^\times}^{\mathrm{GL}_{2n}(\mathbb{F}_q)}(\chi \times \chi)$  and  $R_{\mathbb{F}_q^\times}^{\mathrm{GL}_{2n}(\mathbb{F}_q)}(\chi \circ \mathrm{Nm})$  are orthogonal but have some components in common because  $(T_1, \theta_1)$  and  $(T_2, \theta_2)$  are geometrically conjugate as we check below. By the orthogonality relations for the Deligne-Lusztig representations, both are sums of two distinct representations. The first one of them is  $St_2(\pi) + Id_2(\pi)$  so the only choice for the latter one is  $St_2(\pi) - Id_2(\pi)$ .

Now we check that  $(T_1, \theta_1)$  and  $(T_2, \theta_2)$  are geometrically conjugate. Both the pairs,  $(\mathbb{F}_q^\times \times \mathbb{F}_q^\times, \chi \times \chi)$  and  $(\mathbb{F}_q^\times, \chi \circ \mathrm{Nm})$  give, after the base change to  $\mathbb{F}_{q^{2n}}$ , the same character  $(\chi, \chi^\sigma, \dots, \chi^{\sigma^{n-1}}, \chi, \chi^\sigma, \dots, \chi^{\sigma^{n-1}})$  of  $[(\mathbb{F}_q^n \oplus \mathbb{F}_q^n) \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{2n}}]^\times = [\oplus_{i=1}^n \mathbb{F}_{q^{2n}} + \oplus_{i=1}^n \mathbb{F}_{q^{2n}}]^\times$ , and of  $[\mathbb{F}_{q^{2n}} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{2n}} = \oplus_{i=1}^{2n} \mathbb{F}_{q^{2n}}]^\times$ . Therefore,  $(T_1, \theta_1)$ ,  $(T_2, \theta_2)$  are geometrically conjugate. □



## 6. UNIPOTENT CUSPIDAL REPRESENTATIONS

*Unipotent representations* are constituents of  $R_T^G 1$  for maximal tori  $T$  in  $G/\mathbb{F}_q$ . These are the building blocks of all the representations of  $G(\mathbb{F}_q)$ . Irreducible constituents of  $\text{Ind}_B^{G(\mathbb{F}_q)} 1$  are examples of unipotent representations. These are parametrized by representations of relative Weyl group  $W$  of  $G$  (which we recall is defined to be  $N(T)(\mathbb{F}_q)/T(\mathbb{F}_q)$  for  $T$  a maximal torus of  $G$  which is maximally split). More precisely,  $\text{Ind}_B^{G(\mathbb{F}_q)} 1 = \sum_{\alpha \in \widehat{W}} d(\alpha) \pi_\alpha$ , where  $d(\alpha)$  is the dimension of the irreducible representation  $\alpha$  of  $W$ .

Cuspidal unipotent representations are unipotent representations which are also cuspidal.

**Theorem 6.1.** *For  $G = \text{GL}_n(\mathbb{F}_q)$  there are no cuspidal unipotent representations except for  $n = 1$ . For  $G = \text{U}_n(\mathbb{F}_q)$ , there is a cuspidal unipotent representation if and only if  $n$  is a triangular number, i.e.,  $n = 1 + 2 + \dots + m$  for some  $m$ .*

For classical groups, these representations exist once in a while, whereas for exceptional groups there are plenty of them.

**Example 6.2** (Construction of cuspidal unipotent representation for  $\text{U}_3(\mathbb{F}_q)$ ). If  $\text{U}_3$  is defined by  $z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3$ , let  $T = K^1 \times K^1 \times K^1$ , acting on each  $z_i$  by  $t_i z_i$ . Then  $R_T^{\text{U}_3} 1 = -1 - 2[(q^2 - q)] + St_3$ . The middle term, of dimension  $q^2 - q$ , is (twice) a cuspidal unipotent representation as we check now.

To analyze  $R_T^{\text{U}_3} 1$ , note that by the inner product formula, we have  $\langle R_T^{\text{U}_3} 1, R_T^{\text{U}_3} 1 \rangle = |W| = 6$ .

Let  $T_0 = K \times K^1$  be the maximal torus contained in a Borel subgroup of  $\text{U}_3(\mathbb{F}_q)$ . Then one has  $\langle R_T^{\text{U}_3} 1, R_{T_0}^{\text{U}_3} 1 \rangle = 0$ , where  $R_{T_0}^{\text{U}_3} = 1 + St_3$ . By generalities,  $R_T^{\text{U}_3} 1$  always contains 1 and  $St$ . Thus, by these relations, the only options left for  $R_T^{\text{U}_3} 1$  is  $\pm(1 - St_3 + 2\pi)$ .

We now prove that  $\pi$  is cuspidal. By Proposition 6.3 below, the Jacquet module of  $R_T^{\text{U}_3} 1$  (with respect to a Borel subgroup) is zero, but the Jacquet modules of 1 and  $St_3$  are the same so the Jacquet module of  $\pi$  must be zero.

We finally fix the sign in  $R_T^{\text{U}_3} 1 = \pm(1 - St_3 + 2\pi)$ . For this, one computes the dimension

$$\dim R_T^{\text{U}_3} = \frac{|\text{U}_3|_{p'}}{(q+1)^3} = (q+1)(q^2-1)(q^3+1)/(q+1)^3 = q^3 - 2(q^2 - q) - 1,$$

so the sign must be negative, i.e.,  $R_T^{\text{U}_3} 1 = -(1 - St_3 + 2\pi)$ .

**Proposition 6.3.** *The virtual representation  $R_T^G \theta$  is cuspidal in the sense that all its Jacquet modules are zero if and only if  $T$  is anisotropic.*

*Proof.* One part follows from the transitivity of Deligne-Lusztig induction: if  $T$  is isotropic, then  $T \subset M$ , for a proper Levi subgroup  $M$  of  $G$ , in which case,  $R_G(T, \theta) = R_G(M, R_M(T, \theta))$ , and therefore no irreducible component of  $R_G(T, \theta)$  can be cuspidal.

For the other part, note that  $\langle R_T^G \theta, R_P^G \rho \rangle = 0$  if  $T$  is anisotropic. □

## 7. THE STEINBERG REPRESENTATION

The *Steinberg representation* is an irreducible representation of  $G(\mathbb{F}_q)$  of  $\dim = q^n$  where  $n = \dim U$  where  $U$  is the unipotent radical of a Borel subgroup of  $G$ . It plays a ubiquitous role in all of representation theory. It is defined as follows:

Fix a Borel subgroup  $B_0 \subseteq G$ , then  $St_G = \sum (-1)^{l(P)} \text{Ind}_{P \supseteq B_0}^G 1$  where  $l(P)$  is the number of simple roots in the Levi of the parabolic  $P$ . Equivalently, this is the space of functions on  $B_0(\mathbb{F}_q) \backslash G(\mathbb{F}_q)$  modulo functions invariant on the right by a bigger parabolic.

For any rank 1 group, such as  $\text{GL}_2(\mathbb{F}_q)$  or  $\text{U}_3(\mathbb{F}_q)$ , the Steinberg representation is realized on the space of functions on  $B_0(\mathbb{F}_q) \backslash G(\mathbb{F}_q)$  modulo constants, because  $G$  is the only parabolic containing  $B_0$  in this case.

Steinberg representation has many interesting properties. For instance,  $\Theta_{St}(g) = 0$  if  $g$  is not semisimple and for semisimple elements  $s \in G(\mathbb{F}_q)$ ,  $\Theta_{St}(s) = \epsilon_{G \in Z_G^\circ(s)} |Z_G^\circ(s)|_p$ . In particular, it is non-zero on

semisimple elements, and for regular semisimple elements this character is  $\pm 1$ . Further, the Jacquet module of the Steinberg representation of  $G$  is the Steinberg representation of the Levi subgroup.

## 8. DIMENSION OF THE DELIGNE-LUSZTIG REPRESENTATION

In this section as a sample computation we prove that,

$$\dim R(T, \theta) = \epsilon_G \epsilon_T \frac{|G(\mathbb{F}_q)|_{p'}}{|T(\mathbb{F}_q)|},$$

a statement which requires several ingredients to prove.

The main geometric input in the proof of this dimension formula will be the Lefschetz fixed point formula. Recall that for any variety over a field  $k$ , there are the  $\ell$ -adic cohomology groups with compact support  $H_c^i(X, \mathbb{Q}_\ell)$  which are finite dimensional vector spaces over  $\mathbb{Q}_\ell$ , and non-zero only for  $i \in [0, 2 \dim X]$ . Any automorphism  $\phi$  of  $X$  operates on these cohomology groups, and the alternating sum of their traces is called the Lefschetz number of  $\phi$ , and denoted by  $\mathcal{L}(\phi, X)$ .

**Theorem 8.1.** *Let  $\phi$  be an automorphism of finite order  $n = n_p \cdot n_{p'}$  on a variety  $X$  over a field of characteristic  $p$ . Decompose  $\phi = \phi_s \phi_u$  as a product of automorphisms  $\phi_s = \phi^{a \cdot n_p}$  of order coprime to  $p$ ,  $\phi_u = \phi^{b \cdot n_{p'}}$  of order a power of  $p$  where  $a, b$  are integers with  $an_p + bn_{p'} = 1$ . Then we have the equality of the Lefschetz numbers,*

$$\mathcal{L}(\phi, X) = \mathcal{L}(\phi_u, X^s),$$

where  $X^s$  is the sub-variety of  $X$  on which  $\phi_s$  acts trivially; in particular, if the order of  $\phi$  is coprime of  $p$ , then,

$$\mathcal{L}(\phi, X) = \mathcal{L}(Id, X^\phi).$$

If we are in the context of fixed point free action, then  $X^s$  is empty, giving the following corollary, which is what we would use later.

**Corollary 8.2.** *Let  $\phi$  be an automorphism of finite order prime to  $p$  of a variety  $X$  over a field of characteristic  $p$  which operates without fixed points on  $X$ . Then  $\mathcal{L}(\phi, X) = 0$ .*

We next define a certain notion of duality on the Grothendieck group of representations of  $G(\mathbb{F}_q)$  of finite length which has been independently observed by many people (in different contexts!), to which the names of Alvis, Aubert, Curtis, Deligne, Lusztig, Schneider, Stuhler, Zelevinsky is associated. (Twisting by the sign character seems like a good analogue of this for Weyl groups, in particular for symmetric groups.)

**Theorem 8.3.** *Let  $D_G(\pi) = \sum_{P \supseteq B_0} (-1)^{l(P)} \text{Ind}_P^G(\pi_N)$ . Then*

- (1)  $D_G^2 = Id$ .
- (2)  $D_G$  is an isometry on the Grothendieck group of representations of  $G(\mathbb{F}_q)$ , and as a consequence,  $D_G$  takes irreducible representations to irreducible representations up to a sign.
- (3)  $D_G$  commutes with parabolic induction, and also with Deligne-Lusztig induction.
- (4)  $D_G$  flips 1 and  $St_G$

On cuspidal representations,  $D_G$  is identity, up to a sign.

Note that  $\langle R_T^G 1, 1 \rangle = \epsilon_G \epsilon_T$  and hence by applying  $D_G$  we get that  $\langle R_T^G 1, St \rangle = \epsilon_G \epsilon_T$ . We have  $\langle R_T^G \theta, St \rangle = 0$  if  $\theta \neq 1$  since by the theorem about *geometric conjugacy*, the virtual representations  $R_T^G 1$  and  $R_T^G \theta$  have no irreducible representations in common.

Therefore  $\langle H_c^*(X_U), St \rangle = \langle \sum_\theta R_T^G \theta, St \rangle = \epsilon_G \epsilon_T$ . On the other hand, since the character of the Steinberg representation is supported only on semisimple elements, therefore only on elements of order coprime to  $p$ , the inner product  $\langle H_c^*(X_U), St \rangle$  is calculated by a sum only over elements of order coprime to  $p$ , on which, unless the element is the trivial element, the character of  $H_c^*(X_U)$  is zero by corollary 8.2 (the action of  $G(\mathbb{F}_q)$  on  $X_U$  being given by translation is fixed point free).

Therefore,  $\sum_s \text{semisimple} \Theta_{H_c^*(X_U)}(s) \Theta_{St}(s^{-1}) = |T(\mathbb{F}_q)| \dim R_T^G(\theta) q^{\dim U}$ . This allows us to calculate  $\dim R_T^G \theta$  as desired.

## 9. THE JORDAN DECOMPOSITION OF REPRESENTATIONS

We have defined the notion of a unipotent representation of  $G(\mathbb{F}_q)$ . There is also the notion of a semisimple representation which as a first approximation is defined to be those irreducible representations of  $G(\mathbb{F}_q)$  whose dimension is coprime to  $p$ . (This a good definition only for “good” primes for  $G$ ; in general, one defines an irreducible representation to be semisimple if the average of its character values on regular unipotent elements is nonzero, in which case the average is known to  $\pm 1$ .)

Lusztig has constructed a Jordan decomposition for representations of  $G(\mathbb{F}_q)$  which gives a bijection between the set of irreducible representations  $\pi$  of  $G(\mathbb{F}_q)$ , and pairs  $(\pi_s, \pi_u)$  of irreducible representations in which  $\pi_s$  is a semisimple irreducible representation of  $G(\mathbb{F}_q)$ , and  $\pi_u$  is a unipotent representation of  $Z_{\widehat{G}}(s_\pi)$ , where  $s_\pi$  is a semisimple element associated to  $\pi$  in the dual group  $\widehat{G}$ . The Jordan decomposition has the key property that,

$$\dim \pi = \dim \pi_s \cdot \dim \pi_u.$$

We discuss some of this in a little more detail beginning with the notion of geometric conjugacy of the pair  $(T, \theta)$  which has a nice interpretation in terms of the Langlands dual group  $\widehat{G}$  associated to the reductive group  $G$ . We will not go into details of the dual group here, but emphasize that we take  $\widehat{G}$  to be defined over the same field as  $G$ . For  $G = \text{GL}_n(\mathbb{F}_q), \text{U}_n(\mathbb{F}_q), \text{SO}_{2n}(\mathbb{F}_q)$ , the group  $\widehat{G}$  can be taken to be  $G$  itself, and for  $G = \text{Sp}_{2n}(\mathbb{F}_q)$ ,  $G^\vee = \text{SO}_{2n+1}(\mathbb{F}_q)$ .

**Theorem 9.1.** (1) *To each pair  $(T, \theta)$ , there corresponds a well-defined semisimple conjugacy class  $[s]$  in  $\widehat{G}(\mathbb{F}_q)$ .*  
 (2) *Two pairs  $(T_1, \theta_1)$  and  $(T_2, \theta_2)$  are geometrically conjugate if and only if the associated elements  $s_1$  and  $s_2$  are conjugate in  $\widehat{G}(\overline{\mathbb{F}}_q)$ .*  
 (3) *The correspondence  $(T, \theta) \mapsto [s]$  gives a bijective correspondence between geometric conjugacy classes of pairs  $(T, \theta)$  and semisimple conjugacy classes in  $\widehat{G}(\mathbb{F}_q)$  up to conjugacy in  $\widehat{G}(\overline{\mathbb{F}}_q)$  which is also called geometric conjugacy in  $\widehat{G}(\mathbb{F}_q)$ .*

For a semisimple element  $s \in \widehat{G}(\mathbb{F}_q)$ , define the *Lusztig series*  $\mathcal{E}(G, s)$  to be the union of constituents of  $R_T^G \theta$  such that  $(T, \theta)$  corresponds to an element in the geometric conjugacy class defined by  $s$ . As  $s$  varies in  $\widehat{G}(\mathbb{F}_q)$  up to geometric conjugacy, the Lusztig series  $\mathcal{E}(G, s)$  gives rise a disjoint partition of the set of all irreducible representations of  $G(\mathbb{F}_q)$ . An important example of the Lusztig series is  $\mathcal{E}(G, 1)$  which is the set of all irreducible unipotent representations of  $G(\mathbb{F}_q)$ .

The following theorem is called the *Jordan decomposition of representations*. Before we state the theorem, note that for a semisimple element  $s \in \widehat{G}(\mathbb{F}_q)$ , the centralizer of  $s$  in  $\widehat{G}(\overline{\mathbb{F}}_q)$ , denoted by  $Z_{\widehat{G}}(s)$ , is a reductive group defined over  $\mathbb{F}_q$ . It is known that if the center of  $G$  is connected, such as for an adjoint group, then  $Z_{\widehat{G}}(s)$  is a connected reductive group. We will assume this to be the case in the next theorem, although Deligne-Lusztig theory has an extension to disconnected reductive groups too.

**Theorem 9.2.** *There exists a natural bijection between  $\mathcal{E}(G, s)$  and  $\mathcal{E}(Z_{\widehat{G}}(s), 1)$ ,  $\rho \mapsto \chi_\rho$  which when extended linearly to Grothendieck group of representations takes the Deligne-Lusztig representation  $R_G(T, \theta)$  to  $R_{Z_{\widehat{G}}(s)}(\widehat{T}, 1)$  up to a sign, and has the property that*

$$\dim \rho = \frac{|G|_{p'}}{|Z_{\widehat{G}}(s)|_{p'}} \dim \chi_\rho.$$

**Example :** Observe that since there is no difference between conjugacy and geometric conjugacy in  $\text{GL}_d(\mathbb{F}_q)$ , although in general the Lusztig series  $\mathcal{E}(G, s)$  is a collection of irreducible representations appearing in several  $R(T, \theta)$ , in the case of  $\text{GL}_d(\mathbb{F}_q)$ ,  $\mathcal{E}(G, s) = R(T, \theta)$ , and in fact, it is the set of irreducible

components of a principal series representation of  $\mathrm{GL}_d(\mathbb{F}_q)$  induced from a cuspidal representation of a Levi subgroup.

We now use the Jordan decomposition to work out the dimensions of irreducible representations of  $\mathrm{GL}_{mn}(\mathbb{F}_q)$  appearing in the parabolically induced representation  $\pi \times \cdots \times \pi$  for  $\pi$  a cuspidal representation of  $\mathrm{GL}_m(\mathbb{F}_q)$ .

In this case,  $\widehat{G}(\mathbb{F}_q) = \mathrm{GL}_{mn}(\mathbb{F}_q)$ , and we take  $s = t \times \cdots \times t$  to be a block diagonal matrix with  $t$  a regular elliptic element in  $\mathrm{GL}_m(\mathbb{F}_q)$  represented by a regular element of  $\mathbb{F}_{q^m}^\times$ . The centralizer of  $s$  in  $\mathrm{GL}_{mn}(\mathbb{F}_q)$  is  $\mathrm{GL}_n(\mathbb{F}_{q^m}) \hookrightarrow \mathrm{GL}_{mn}(\mathbb{F}_q)$ . Thus the irreducible representations appearing in the principal series representation  $\pi \times \cdots \times \pi$  for  $\pi$  a cuspidal representation of  $\mathrm{GL}_m(\mathbb{F}_q)$ , are parametrized by the unipotent representations of  $\mathrm{GL}_n(\mathbb{F}_{q^m})$ , which are parametrized by the partitions of  $n$ ; further, the ratios of the dimensions of irreducible representations appearing in the principal series representation  $\pi \times \cdots \times \pi$  is the same as the ratios of the dimensions of irreducible representations appearing in the principal series representation  $\mathrm{GL}_n(\mathbb{F}_{q^m})$  induced from the trivial character of the Borel; this generalizes the conclusion of Proposition 5.1.

## 10. EXERCISES

### 10.1. Algebraic groups.

- (1) If  $K/k$  is a degree  $d$  extension of a field  $k$  then  $K^\times \subseteq \mathrm{GL}_d(k)$ .
- (2) Prove that  $|\mathrm{U}_n(\mathbb{F}_q)| = |\mathrm{GL}_n(\mathbb{F}_{-q})|$ .
- (3) For a torus  $T/\mathbb{F}_q$ , we have  $|T(\mathbb{F}_q)| = 1$  if and only if  $q = 2$  and  $T$  is split.
- (4) Reconcile the classification of classical groups discussed in section 2 in terms of algebras with involutions with what you know about forms of  $\mathrm{GL}_n, \mathrm{U}_n, \mathrm{SO}_n, \mathrm{Sp}_n$ .
- (5) For  $\prod K_i^\times \hookrightarrow \mathrm{GL}(\bigoplus K_i)$  calculate the Weyl group.
- (6) For the tori constructed inside  $T \hookrightarrow G(A, *) \hookrightarrow \mathrm{Aut}(A)$ , calculate the Weyl group of  $T$  inside  $G(A, *)$  and compare the answer with the Weyl group of  $T$  inside  $\mathrm{Aut}(A)$ .
- (7) (Root system) Let  $\mathrm{U}_n(k)$  be defined by the hermitian form

$$\begin{pmatrix} & & & & 1 \\ & & & & \\ & & & 1 & \\ & & \cdots & & \\ & & & & \\ 1 & & & & \end{pmatrix},$$

i.e.,  $H(\vec{z}, \vec{w}) = \sum z_i \bar{w}_{n+1-i}$ . Let

$$T = \begin{pmatrix} t_1 & & & & & & \\ & \cdots & & & & & \\ & & t_m & & & & \\ & & & t_m^{-1} & & & \\ & & & & \cdots & & \\ & & & & & & t_1^{-1} \end{pmatrix}, \quad T = \begin{pmatrix} t_1 & & & & & & \\ & \cdots & & & & & \\ & & t_m & & & & \\ & & & 1 & & & \\ & & & & t_m^{-1} & & \\ & & & & & \cdots & \\ & & & & & & t_1^{-1} \end{pmatrix}$$

be the respective maximal split tori for  $n = 2m$  and for  $n = 2m + 1$  with  $t_i \in k^\times$ . Calculate the root space decomposition.

- (8) Identify the maximal tori in  $\mathrm{Sp}_4(\mathbb{F}_q)$  up to conjugacy. Are there isomorphic but non-conjugate tori? Are there isomorphic but non-stably conjugate tori? (Two tori  $T_1$  and  $T_2$  are said to be stably conjugate in  $G(k)$  if there exists  $g \in G(\bar{k})$  taking  $T_1$  to  $T_2$  by inner-conjugation which as a homomorphism from  $T_1$  to  $T_2$  is defined over  $k$ .)

### 10.2. Representation theory.

- (1) Classify representations of  $\mathrm{GL}_2(\mathbb{F}_q)$  which do not remain irreducible when restricted to  $\mathrm{SL}_2(\mathbb{F}_q)$ . How many are there?

- (2) The principal series  $Ps(\chi_1, \chi_2)$  of  $GL_2(\mathbb{F}_q)$  is a sum of 2 irreducibles when restricted to  $SL_2$  if and only if  $\chi_1\chi_2^{-1}$  is a nontrivial character of order 2.
- (3) Classify those principal series representations of  $SL_2(\mathbb{F}_q)$  which are reducible.
- (4) Classify irreducible representations of  $GL_2(\mathbb{F}_q)$  which have a self-twist, i.e.,  $\pi = \pi \otimes \chi$ ,  $\chi \neq 1$ .
- (5) Identify the unique cuspidal representation of  $GL_2(\mathbb{F}_2) \cong S_3$ .
- (6) Identify the degrees of the cuspidal representations of  $SL_2(\mathbb{F}_4) \cong A_5$ .
- (7) Identify distinct conjugacy classes in  $SL_2(\mathbb{F}_q)$  which become the same in  $GL_2(\mathbb{F}_q)$ .
- (8) Identify the affine curve  $XY^q - X^qY = 1$  over  $\mathbb{F}_q$  to be the Deligne-Lusztig variety  $X_U$  defined in these notes associated to a non-split torus in  $SL_2(\mathbb{F}_q)$ .
- (9) Identify the automorphism group of the curve  $XY^q - X^qY = 1$  over  $\mathbb{F}_q$ .
- (10) In Theorem 4.2 prove that  $u \in GL_m(\mathbb{F}_{q^d})$ ,
- (11) Calculate the dimensions of the two components of  $\rho \times \rho$  for  $\rho$  cuspidal of  $GL_2(\mathbb{F}_q)$ .
- (12) Given a unipotent element  $u_\lambda$  in  $GL_n$  for a partition  $\lambda$  of  $n$ , calculate the number of Borel subgroups such that  $u_\lambda \in B$ .
- (13) Calculate:  $\text{Ind}_B^{GL_3(\mathbb{F}_q)} 1$  as a sum of irreducibles and verify that we have one irreducible constituent of dimension 1, two of dimension  $(q^2 - q)$  and one of dimension  $q^3$ , the Steinberg.
- (14) Do the above exercise for  $U_3(\mathbb{F}_q)$ .

### 10.3. Further Exercises.

- (1) Prove that there exists a natural bijective correspondence between representations of  $GL_n(\mathbb{F}_q)$  over  $\overline{\mathbb{Q}}_l^\times$ , and conjugacy classes in  $GL_n(\mathbb{F}_q)$  after having made the choice of an embedding  $\overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}_l^\times$ .
- (2) Prove that there is an isomorphism:  $\varinjlim X^*(\mathbb{F}_{q^n}^\times) \cong \overline{\mathbb{F}}_q^\times$  as  $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ -modules where the groups  $X^*(\mathbb{F}_{q^n})$  form a directed system of groups via the norm mappings:  $\mathbb{F}_{q^d}^\times \rightarrow \mathbb{F}_{q^{d'}}^\times$  for  $d'|d$ . (Hint: For any cyclic group  $C$  of order  $d$ , the automorphism group of  $C$  is canonically isomorphic to  $(\mathbb{Z}/d)^\times$ .)

## 11. REFERENCES

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