SOME REMARKS ON REPRESENTATIONS OF QUATERNION DIVISION ALGEBRAS

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Abstract. For the quaternion division algebra D over a non-Archimedean local field k, and π an irreducible finite dimensional representation of $D^\times$, say with trivial central character, we prove the existence of a quadratic extension K of k such that the trivial character of $K^\times$ appears in π, as well as the existence of a quadratic extension L of k such that the trivial character of $L^\times$ does not appear in π. Consequences for theta lifts, and variation of local root numbers under quadratic twists, are made.

The aim of this paper is to make some remarks on representations of $D^\times$ restricted to tori where D is a quaternion division algebra over a non-Archimedean local field k, and deduce some consequences for variation of local root numbers under quadratic twists using theorems of Saito and Tunnell. By the Jacquet-Langlands correspondence, these remarks can also be made for discrete series representations of $\mathrm{GL}_2(k)$.

Our first result, motivated by some applications to theta correspondence to which we come to later, should be contrasted with the situation for $\mathrm{SU}_2(\mathbb{R})$ where for irreducible representations π_1 and π_2 of $\mathrm{SU}_2(\mathbb{R})$ with the same central characters, any character of the maximal torus $\mathbb{S}^1 \hookrightarrow \mathrm{SU}_2(\mathbb{R})$ that appears in π_1 also appears in π_2 if $\dim \pi_1 \leq \dim \pi_2$. We prove that this does not happen for non-Archimedean local fields. It may be of interest to investigate a similar question for a general compact connected Lie group G with a maximal torus T: what are irreducible representations π_1 and π_2 of G such that each character of T appearing in π_1 also appears in π_2 (counted with multiplicity, so that $\pi_1|_T \subset \pi_2|_T$).

Proposition 0.1. *Let π_1 and π_2 be two irreducible representations of $D^\times$ where D is the quaternion division algebra over a non-Archimedean local field k of odd residue characteristic with the same central characters. Assume that $\dim(\pi_1) \leq \dim(\pi_2)$. Then there exists a quadratic extension K of k, and a character $\chi : K^\times \to \mathbb{C}^\times$ which appears in π_1 but not in π_2 unless $\pi_1 = \pi_2$.*

The proof of this proposition will be based on a few preliminary results. The next proposition is about certain multiplicity one results for which we refer to [P].

Proposition 0.2.
1. *Let π be an irreducible representation of $D^\times$, and K a quadratic extension of k. Then any character $\chi : K^\times \to \mathbb{C}^\times$ appears in π with multiplicity at most 1.*
2. *Let π_1, π_2, π_3 be three irreducible representations of $D^\times$, then the space of $D^\times$-invariant linear forms $\ell : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ has dimension at most 1.*

Before we state the next proposition, we introduce some notation. Let $\mathcal{O}$ be the maximal compact subring of k, $\mathcal{O}_D$ the maximal compact subring of D, π_D an element of $\mathcal{O}_D$ such that $\pi_D\mathcal{O}_D$ is its unique maximal ideal with $\mathcal{O}_D/\pi_D\mathcal{O}_D$ a finite field $\mathbb{F}_{q^2}$

which is a degree 2 extension of $\mathbb{F}_q$ which is the residue field of k. The element π_D is called a uniformizing parameter of $\mathcal{O}_D$. There is a natural filtration $D^\times(m)$ of $D^\times$ for $m \geq 0$ with $D^\times(0) = \mathcal{O}_D^\times$, and $D^\times(m) = 1 + \pi_D^m \mathcal{O}_D$ for $m > 0$. We have $D^\times(m)/D^\times(m+1) \cong \mathbb{F}_{q^2}$ for $m > 0$. Similarly, for any extension K of k, there is a natural filtration $K^\times(m)$ of $K^\times$, but note that if K is a subfield of D, then the two filtrations $K^\times(m)$ and $D^\times(m)$ need not correspond.

Define the level of a representation π of $D^\times$ to be the minimum integer $n \geq 0$ such that π restricted to $D^\times(n)$ is trivial. For a representation π of $D^\times$, and a character ω of $k^\times$, there is the notion of twisting π by ω, denoted $\pi \otimes \omega$. An irreducible representation π of $D^\times$ is said to be of minimal level if one cannot decrease the level of π by twisting by a character, i.e., level$(\pi \otimes \omega) \geq$ level(π) for all characters ω of $k^\times$. The minimal level of a representation π is the minimum among the integers level$(\pi \otimes \omega)$.

The following proposition (as well as its analogue for any division algebra of prime index) is there in the works of Carayol, Howe, and Tunnell among others; cf. Proposition 6.5 of [Ca].

Proposition 0.3. *Suppose π is an irreducible representation of $D^\times$ of minimal level n. Then the dimension of π depends only on n. If $n = 2m$, the dimension of π is $q^{m-1}(q+1)$, and if $n = 2m+1$, the dimension of π is $2q^m$.*

Proof of Proposition 1. If π_1 and π_2 have the same dimension, and if the characters of $K^\times$ appearing in π_1 are the same as the characters of $K^\times$ appearing in π_2 for any quadratic extension K of k, then by character theory $\pi_1 \cong \pi_2$. We therefore assume that π_1 and π_2 have different dimensions; thus $\dim(\pi_1) < \dim(\pi_2)$. We will assume that every character of $K^\times$ which appears in π_1 also appears in π_2 for any quadratic extension K of k, and derive a contradiction from this.

Let $\pi_2^\vee$ denote the contragredient of π_2, and look at the representation $\pi_1 \otimes \pi_2^\vee$. Since every character of $K^\times$ which appears in π_1 appears in π_2, it follows that the trivial character of $K^\times$ appears with multiplicity $= \dim(\pi_1)$ in $\pi_1 \otimes \pi_2^\vee$.

Since $\dim(\pi_1) < \dim(\pi_2)$, in particular by proposition 3 the minimal level of π_1 is less than that of π_2, and hence $\pi_1 \otimes \pi_2^\vee$ is a sum of $\dim(\pi_1)$ many (distinct, by part (2) of Proposition 2) irreducible representations of $D^\times/k^\times$, all of the same minimal level as that of π_2. (This conclusion about $\pi_1 \otimes \pi_2^\vee$ was also obtained in Proposition 7.2 of [P1] but there it was proved only in odd residue characteristic.) By part (1) of proposition 2, any irreducible representation of $D^\times$ contains the trivial representation of $K^\times$ with multiplicity at most 1; it follows that every irreducible component of $\pi_1 \otimes \pi_2^\vee$ contains the trivial representation of $K^\times$ for any quadratic extensions K of k.

The following proposition now completes the proof of our main proposition (where we need only part 1 for application above).

Proposition 0.4. *Let π be an irreducible representation of $D^\times$ with central character $\omega_\pi = \chi^2$ where χ is a certain character of $k^\times$. Then the following holds.*

1. *If π is not 1 dimensional, then there exists a quadratic extension K of k, if k has odd residue characteristic, such that the restriction of π to $K^\times$ does not contain the character of $K^\times$ given by $\chi \circ \mathrm{Nm} : K^\times \to \mathbb{C}^\times$.*
2. *There exists a quadratic extension K of k such that the restriction of π to $K^\times$ contains the character of $K^\times$ given by $\chi \circ \mathrm{Nm} : K^\times \to \mathbb{C}^\times$. More precisely,*

if the minimal level of π is odd, then this conclusion holds for all quadratic ramified K, whereas if the minimal level of π is even, then this conclusion holds for the quadratic unramified K.

Proof. We begin by noting that both the parts of the proposition are invariant under twisting by characters, and therefore we will always assume during the course of this proof that the representation π is of minimal level.

The proof of the first part of the proposition depends on the observation that if N is a normal subgroup of $D^\times$ then π restricted to N is a sum of irreducible representations of N which are all conjugate under the inner-conjugation action of $D^\times$ on N; in particular, if π restricted to N contains the trivial representation of N, then N operates trivially on the representation space of π. This implies that if π restricted to N contains the restriction to N of a character of $D^\times$ (such as $\chi \circ \mathbb{N}\mathrm{m}$), then π restricted to N is a multiple of this character of $D^\times$ restricted to N, and therefore $\pi \otimes \chi$ is trivial on N. If π is supposed to be minimal, this cannot happen if $N = D^\times(m)$ and the level of π is larger than m.

Suppose that $n > 0$ is the level of the representation π. If the level of π is odd, say $n = 2m+1$, then $D^\times(2m)/D^\times(2m+1)$ is represented by elements of $K_u^\times$ where K_u is the quadratic unramified extension of k. Since π is a minimal irreducible representation, as noted in the previous paragraph, π restricted to $D^\times(2m)$ cannot contain the restriction of the character $\chi\circ\mathbb{N}\mathrm{m}$ to $D^\times(2m)$, hence a fortiori π restricted to $K_u^\times$ does not contain the character $\chi \circ \mathbb{N}\mathrm{m}$ of $K_u^\times$.

Assume now that π has even level, say $n = 2m$. In this case π is a representation of $D^\times/D^\times(2m)$ which is nontrivial restricted to the normal subgroup $D^\times(2m-1)/D^\times(2m) \cong \mathbb{F}_{q^2}$. On this normal subgroup $D^\times/D^\times(2m)$ operates by conjugation through the quotient $D^\times/k^\times D^\times(1)$. We note that $\mathcal{O}_D^\times/\mathcal{O}^\times D^\times(1) = \mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$, which can be identified to the group $\mathbb{S}^1(\mathbb{F}_q)$ of norm 1 elements of $\mathbb{F}_{q^2}^\times$, and the associated action of $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times = \mathbb{S}^1(\mathbb{F}_q)$ on $D^\times(2m-1)/D^\times(2m) \cong \mathbb{F}_{q^2}$ is via multiplication of $\mathbb{S}^1(\mathbb{F}_q)$ on $\mathbb{F}_{q^2}$; furthermore, the action of π_D on $D^\times(2m-1)/D^\times(2m) \cong \mathbb{F}_{q^2}$ is the natural Galois action of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ on $\mathbb{F}_{q^2}$.

As is well-known, one can identify the character group of $\mathbb{F}_{q^2}$ with $\mathbb{F}_{q^2}$ on which $\mathbb{S}^1(\mathbb{F}_q)$ operates in the usual way by multiplication.

We will now use the observation that if there are less than $q+1$ linear forms on the 2 dimensional $\mathbb{F}_q$-vector space $\mathbb{F}_{q^2}$, there is a one dimensional subspace of $\mathbb{F}_{q^2}$ on which none of these linear forms is trivial. Let $\mathbb{P}^1(\mathbb{F}_q)$ be the projective line of one dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^2}$. (The $\mathbb{F}_{q^2}$ appearing here is the character group of $D^\times(2m-1)/D^\times(2m)$.)

Claim: $D^\times$ is not transitive in its action on one dimensional $\mathbb{F}_q$-subspaces of $D^\times(2m-1)/D^\times(2m) \cong \mathbb{F}_{q^2}$. For this, note that there is a natural surjective map from $\mathbb{P}^1(\mathbb{F}_q)$ to $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$ given by sending a line in $\mathbb{F}_{q^2}$ generated by a vector v to $\mathbb{N}\mathrm{m}(v) \in \mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$. Since $\mathbb{N}\mathrm{m}(v) = \mathbb{N}\mathrm{m}(\bar{v})$, where $\bar{v}$ is the Galois conjugate of v in $\mathbb{F}_{q^2}$, and $\mathbb{S}^1(\mathbb{F}_q)$ preserves norms, $D^\times$ acts on $\mathbb{P}^1(\mathbb{F}_q)$ preserving the norm to $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$, and therefore the action is not transitive on $\mathbb{P}^1(\mathbb{F}_q)$. (It is in this part of the proof, we are using that $-1 \neq 1 \in \mathbb{F}_q$.)

It follows that there is a copy of $\mathbb{F}_q$ inside $\mathbb{F}_{q^2}$ considered as a two dimensional vector space over $\mathbb{F}_q$ such that none of the characters of $D^\times(2m-1)/D^\times(2m) \cong \mathbb{F}_{q^2}$ appearing in the restriction of π to this subgroup are trivial on this $\mathbb{F}_q$. We note that the character

$\chi \circ \mathbb{N}\mathrm{m}$ of $D^\times$ restricted to $D^\times(2m-1)$ is the trivial character. Thus we conclude that there is a ramified torus $K^\times$ inside $D^\times$ such that the restriction of π to $K^\times$ does not contain the character $\chi \circ \mathbb{N}\mathrm{m}$.

We now turn to part (2) of the proposition. We assume as in part (1) that the representation π has minimal level. Now, part (2) of the proposition follows from the multiplicity 1 result about characters of $K^\times$ appearing in irreducible representations of $D^\times$ when combined with Proposition 3 about dimension of irreducible representations. To wit, if the level of the representation is even, say $n = 2m$, then π restricted to $K_u^\times$ has exactly those characters of $K_u^\times$ which are trivial on $K_u^\times(m)$ and whose restriction to $k^\times$ is the central character of π. The cardinality of the set of such characters is $(q+1)q^{m-1}$ which is the same as dimension of π, so by multiplicity 1, all characters of $K_u^\times$ whose restriction to $K_u^\times(m)$ is trivial, and whose restriction to $k^\times$ is the central character of π, and in particular the character of $K_u^\times$ given by $\chi \circ \mathbb{N}\mathrm{m} : K_u^\times \to \mathbb{C}^\times$, appears in π.

By a similar reasoning, if the minimal level is odd, then the character of $K^\times$ given by $\chi \circ \mathbb{N}\mathrm{m} : K^\times \to \mathbb{C}^\times$ appears in π for K any ramified quadratic extension of k. $\square$

Remark : Part (2) of Proposition 4 allows one to some kind of a theory of newforms for representations of $D^\times$.

Question: Wee Teck Gan has asked if the conclusion of the 2nd part of the previous proposition holds good for division algebras of higher degree than 2, specially for cubic division algebras. More precisely, let D be a cubic division algebra over a non-Archimedean local field k, and π an irreducible representation of $D^\times/k^\times$. Then is it true that there is a cubic subfield K of D such that π has a vector fixed under $K^\times/k^\times$? See the work of Savin in [Sa] which proves that if this were true then certain theta lifts from $D^\times/k^\times$ to G_2 would be nonzero. In a work of Gan and Gurevich in [GG], they have constructed lifting of representations from inner-forms of GL_3 (both local and global) to G_2; an affirmative answer to this question will show nonvanishing of these lifts (just as the present work is related to nonvanishing of certain theta lifts).

Remark : Apropos the previous remark, it is known that for any adjoint compact connected Lie group G, any irreducible representation of G has a vector fixed by a maximal torus, cf. Proposition 11.3(a) in [Ka]. I would like to thank Shrawan Kumar for pointing this out to me.

1. Application to Theta Correspondence

We use proposition 4 to make some conclusions about the theta correspondence over a non-Archimedean local field k.

Recall that if V is an orthogonal space, and W a symplectic space, then the theta correspondence is a certain correspondence between irreducible representations of $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ (or of the 2 fold cover, the metaplectic group $\widehat{\mathrm{Sp}}(W)$) obtained by looking at the Weil representation of $\widehat{\mathrm{Sp}}(V \otimes W)$ on the Schwartz space of functions on $V \otimes W_-$ where W_- is a maximal isotropic subspace of W.

If V is 3 dimensional, then $\mathrm{SO}(V)$ is isomorphic either to $D^\times/k^\times$, or to $\mathrm{PGL}_2(k)$. If W is 2 dimensional, then Weil representation of $\widehat{\mathrm{Sp}}(V \otimes W)$ is realized on the Schwartz space of functions on V on which $\mathrm{O}(V)$ operates in the natural way. Identifying $\mathrm{SO}(V)$ to $D^\times/k^\times$, or to $\mathrm{PGL}_2(k)$ as the case may be, the action of $\mathrm{SO}(V)$ on V gets translated to the inner conjugation action of $D^\times/k^\times$, or of $\mathrm{PGL}_2(k)$ as the case may be on the trace 0 elements of D (respectively $M_2(k)$). So a corollary of Proposition 4 is the following well-known result:

Proposition 0.5. *Every irreducible admissible representation of $D^\times/k^\times$ as well as every irreducible admissible representation of $\mathrm{PGL}_2(k)$ appears in theta correspondence with $\widehat{SL}_2(k)$.*

One can use the main result of this paper also to prove the following proposition about theta correspondence. This proposition was in fact the *raison d'être* of this note. Before stating the proposition, we introduce some more notation. Let $\mathrm{GSp}_D(4)$ be the rank 1 form of the symplectic similitude group defined using a Hermitian form on a 2 dimensional vector space over D. Similarly define the rank 1 form of $\mathrm{GSO}(4)$ to be denoted by $\mathrm{GSO}_D(4)$, and defined using a skew-Hermitian form on a 2 dimensional vector space over D. It is easy to see that $\mathrm{GSO}_D(4) \cong [D^\times \times \mathrm{GL}_2(k)]/\Delta(k^\times)$ where $\Delta(k^\times)$ is the embedding of $k^\times$ in $D^\times \times \mathrm{GL}_2(k)$ as (t, t^{-1}). The theta correspondence can be defined between $\mathrm{GSO}_D(4)$ and $\mathrm{GSp}_D(4)$.

Proposition 0.6. *The theta lift of an irreducible representation $\pi_1 \boxtimes \pi_2$ of $\mathrm{GSO}_D(4) \cong [D^\times \times \mathrm{GL}_2(k)]/\Delta(k^\times)$ to $\mathrm{GSp}_D(4)$ is nonzero if and only if π_1 and π_2 are not related to each other by the Jacquet-Langlands correspondence.*

Proof. We do not prove this proposition in detail but point out that an irreducible representation of $\mathrm{GSp}_D(4)$ which is not 1 dimensional is nonzero if and only if it has a nonzero Bessel coefficient. Now the χ-Bessel coefficients of $\mathrm{GSp}_D(4)$ are nonzero exactly for those characters χ of $K^\times$ which appear in both π_1 and π_2. Proposition 1 says that there are no such characters if and only if π_1 and π_2 are related to each other by the Jacquet-Langlands correspondence. We refer to [PT] for the notion of Bessel coefficients as well as its relation to characters of tori. □

2. Application to local root numbers

Variation of root numbers (both local and global) under quadratic twists is a popular theme of much current interest. We refer to the article to Tunnell [Tu] as a convenient reference for the definition and basic properties of local root numbers (also called local epsilon factors). We note that these local root numbers $\epsilon(\sigma,\psi)$, for σ a representation of the Weil group W_k, depend on the choice of a nontrivial auxiliary character $\psi : k \rightarrow \mathbb{C}^\times$, but if $\det \sigma = 1$, is in fact independent of ψ, and is hence omitted from the notation. If $\sigma \cong \sigma^\vee$, and $\det \sigma = 1$, then $\epsilon(\sigma) = \pm 1$.

Since by a theorem of Saito and Tunnell, cf. [S] and [Tu], characters of representations of $D^\times$ are related to certain local root numbers, we can use our earlier proposition 4 to make conclusions about variation of local root numbers under quadratic twists. (We will use only the 2nd part of the proposition valid for all non-Archimedean local fields.) The difficult case to handle is naturally the residue characteristic 2 case where there are *exceptional* representations whose local root numbers may not be as easily amenable to a direct calculation.

Proposition 0.7. *Let σ be a two dimensional irreducible representation of determinant 1 of the Galois group of a local field k. Then there exists a quadratic character ω of $k^\times$ such that the epsilon factor $\epsilon(\sigma) \neq \epsilon(\sigma \otimes \omega)$ if either k is of even residue characteristic, or $\sqrt{-1}$ belongs to $k^\times$; if k is of odd residue characteristic, and $\sqrt{-1} \notin k^\times$, then there exists ω a character of $k^\times$ with $\omega^2 = 1$ with $\epsilon(\sigma) \neq \epsilon(\sigma \otimes \omega)$ if and only if the Artin conductor of σ is odd.*

Proof. By generalities about epsilon factors twisted by a quadratic *unramified* character ω,

$$\epsilon(\sigma \otimes \omega) = \omega(\pi_k)^{a(\sigma)} \epsilon(\sigma),$$

where π_k is a uniformizing parameter of $k^\times$, and $a(\sigma)$ is the Artin conductor of σ. Thus if the Artin conductor of σ is odd, the result in the proposition is clear. It suffices to assume then, for the rest of the proof, that the Artin conductor of σ is even.

Let π be the representation of $D^\times/k^\times$ associated to the representation σ of the Galois group of k. By generalities, it is known that the level of the representation π is equal to $a(\sigma) - 1$, which is thus odd in our case. By proposition 4, the trivial character of $K^\times$ appears in π for any quadratic ramified extension K of k. For any quadratic character ω of $k^\times$, let K_ω be the quadratic field defined by ω.

We recall that by a theorem of Saito and Tunnell, the trivial character of $K_\omega^\times$ for K_ω a quadratic extension of k appears in π if and only if $\epsilon(\pi)\epsilon(\pi \otimes \omega) = -\omega(-1)$. Since as we just noted, the trivial character of $K_\omega^\times$ appears in π for all quadratic ramified extensions K_ω of k, for any quadratic ramified character ω, we get

$$\epsilon(\pi)\epsilon(\pi \otimes \omega) = -\omega(-1).$$

Thus if there is a quadratic ramified character ω such that $\omega(-1) = 1$, we will have achieved a change in epsilon under quadratic twist by ω.

We now note that if $\sqrt{-1}$ belongs to $k^\times$, then $\omega(-1) = 1$ for any quadratic character ω of $k^\times$. On the other hand, if $\sqrt{-1}$ does not belong to $k^\times$, then -1 is a non-trivial element of $k^\times/(k^\times)^2$, and hence there is a quadratic character ω of $k^\times$, necessarily a ramified character, such that $\omega(-1) = -1$. If k is of odd residue characteristic, then there are exactly 2 ramified characters which differ by a quadratic unramified character. Hence in odd residue characteristic, if $\sqrt{-1}$ does not belong to $k^\times$, then $\omega(-1) = -1$ for both the ramified characters, and hence one cannot change the sign of the epsilon factor $\epsilon(\pi)$ by twisting π. In the even residue characteristic case, there are ramified characters ω_1 and ω_2 such that $\omega_3 = \omega_1\omega_2$ is still ramified. Clearly $\omega_i(-1) = 1$ for at least one i, completing the proof of the proposition. □

If $\epsilon(\sigma) \neq \epsilon(\sigma \otimes \omega)$, then in particular, $\sigma \ncong \sigma \otimes \omega$ for some character ω of W_k with $\omega^2 = 1$. On the other hand hand, there are actually two dimensional representations σ of W_k (necessarily irreducible, and necessarily in odd residue characteristic) for which $\sigma \cong \sigma \otimes \omega$ for all characters ω of W_k with $\omega^2 = 1$. We therefore obtain the following curious corollary to the previous proposition, for which we give a more conventional proof in the following remark.

Corollary 1. *Let k be of odd residue characteristic, and assume that $(-1) \in k^{\times 2}$. Then the determinant of an irreducible representation of W_k which is dihedral with respect to all the three quadratic extensions of k cannot be trivial.*

Remark : Under the homomorphism $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$ (where we take SO_3 defined by the quadratic form $X^2 + Y^2 + Z^2$), the representations in corollary above correspond to homomorphisms from W_k to the group of 3×3 diagonal matrices $(\chi_1(x), \chi_2(x), \chi_3(x))$ with $\chi_i^2 = 1$, and $\chi_1\chi_2 = \chi_3$. The two-fold cover $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{C})$, when restricted to the diagonal subgroup of $\mathrm{SO}_3(\mathbb{C})$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus Z/2\mathbb{Z}$, gives rise to the two fold cover of this group which is nontrivial on any $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/2\mathbb{Z} \oplus Z/2\mathbb{Z}$, which is the quaternionic group Q_8 of order 8. If e_1, e_2, e_3 are the three nontrivial elements of $H^1(\mathbb{Z}/2\mathbb{Z} \oplus Z/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus Z/2\mathbb{Z}$, then the central extension of $\mathbb{Z}/2\mathbb{Z} \oplus Z/2\mathbb{Z}$ defined by Q_8 is the element $e_1 \cup e_2 + e_2 \cup e_3 + e_3 \cup e_1$ in $H^2(\mathbb{Z}/2\mathbb{Z} \oplus Z/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$. Denoting the pull back of these cohomolgy classes to $H^i(W_k, \mathbb{Z}/2\mathbb{Z})$ by the same symbol, we need to calculate $e_1 \cup e_2 + e_2 \cup e_3 + e_3 \cup e_1$ in $H^2(W_k, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Br}_k[2] = \mathbb{Z}/2\mathbb{Z}$. The Hilbert symbol $H = (\cdot, \cdot) : k^\times/k^{\times 2} \times k^\times/k^{\times 2} \to \mathbb{Z}/2\mathbb{Z}$ is a nondegenerate symmetric bilinear form. Therefore if k is a non-Archimedean local field of odd residue characteristic, there are two options:

(1) $(x, x) = 1$ for all $x \in k^\times$. Since $(x, -x) = 1$, $(x, x) = 1$ for all $x \in k^\times$ precisely when -1 is a square in k. Assuming this to be the case, the matrix corresponding to the bilinear form given by H is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in an appropriate basis of $k^\times/k^{\times 2}$.

(2) $(x, x) = -1$ for some $x \in k^\times$, which as we just noted happens precisely when -1 is not a square in $k^\times$. In this case, the matrix corresponding to the bilinear form given by H is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in an appropriate basis of $k^\times/k^{\times 2}$.

It is easy to calculate $(e_1, e_2) + (e_2, e_3) + (e_3, e_1) = (e_1, e_1) + (e_2, e_2) + (e_1, e_2)$ (sum in $\mathbb{Z}/2\mathbb{Z}$) in the two cases, giving us 1 in the first case, and 0 in the second case, in accordance with the corollary! (In fact we get the sharper conclusion that there is a two dimensional irreducible representation of W_k which is dihedral with respect to all the three quadratic extensions of an odd residue characteristic local field k of trivial determinant if and only if $-1 \notin k^{\times 2}$.)

3. Concluding remarks

The phenomenon noted at the beginning of this paper (and in proposition 1) which contrasts Archimedean and non-Archimedean fields seems closely related to the fact that there is more flexibility to local root numbers in the non-Archimedean case, as there is more room in non-Archimedean fields. We take the occasion to formulate a question.

Question : Let K be a quadratic extension of a p-adic field k. Let $\chi_1, \chi_2, \cdots, \chi_n$ be n *distinct* characters of $K^\times$ of *symplectic type*, i.e., for which the restriction to $k^\times$ is the quadratic character ω_K of $k^\times$. Let ψ be a nontrivial character of K trivial on k. For characters χ of $K^\times$ which are trivial on $k^\times$, the characters $\chi_i \cdot \chi$ of $K^\times$ are of symplectic type, and the local root numbers $\epsilon(\chi_i \cdot \chi, \psi)$ take values in $\{\pm 1\}$ for all integers i with $1 \leq i \leq n$. Thus as χ runs over characters of $K^\times/k^\times$, we get an element of $(\pm 1)^n$, which might be called *a pattern* for $\{\chi_i\}$. For a given $\{\chi_i\}$ as above, which patterns in $(\pm 1)^n$ arise in this way? (The next remark shows that if n is large compared to the size of the residue field of k, then not all patterns arise.)

Remark : By a theorem of Deligne, cf. Theorem 1.2 in [Tu], whenever cond$(\beta) > 2$ cond(α),

$$\epsilon(\alpha\beta, \psi) = \alpha(y_\beta)\epsilon(\beta, \psi),$$

for an element y_β of $K^\times$, which if β is trivial on $k^\times$ can be taken to belong to $k^\times$. Therefore if β is a character of $K^\times$ trivial on $k^\times$, by the theorem of Frohlich-Queryut $\epsilon(\beta, \psi) = 1$ (note that ψ is trivial on k), and therefore,

$$\epsilon(\alpha\beta, \psi) = \alpha(y_\beta)\epsilon(\beta, \psi) = \omega_K(y_\beta).$$

It follows that there are (at most) two patterns which repeat infinitely often:

$$\{-, -, \cdots, -, -\},$$

and

$$\{+, +, \cdots, +, +\},$$

Remark : If $K = \mathbb{C}$, and $k = \mathbb{R}$, then the symplectic characters of $\mathbb{C}^\times$ are parametrized by odd integers $n \in \mathbb{Z}$, given by $\chi_n : re^{i\theta} \to e^{in\theta}$. For a suitable character ψ of $\mathbb{C}$ as above, $\epsilon(\chi_n, \psi) = 1$ if $n > 0$, and $\epsilon(\chi_n, \psi) = -1$ if $n < 0$. Therefore if the characters χ_i in the above question are put in an increasing order, it follows that the only patterns possible in the above question are,

$$\{-, -, \cdots, -, +, +, \cdots +\},$$

and therefore, there are exactly $n + 1$ patterns possible in the Archimedean case for the above question, and among these, exactly 2 patterns,

$$\{-, -, \cdots, -, -\},$$

and

$$\{+, +, \cdots, +, +\},$$

correspond to infinitely many choices for the character χ.

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