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Bézout's theorem for abelian varieties

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Abstract. We give a very short and simple proof of Bézout's theorem for simple abelian varieties over an algebraically closed field of characteristic zero.

We prove the following analogue for simple abelian varieties (i.e. an abelian variety not containing any non-zero proper abelian subvariety) of Bézout's theorem for $\mathbf{P}^n$. The base field in this paper will be an algebraically closed field of characteristic 0. After this note was written, we learnt that the result is a known one; see W. Barth: Fortsetzung meromorpher Funktionen in Tori und komplex-projective Räumen, Inv. Math. vol. 5, 42–62(1968). However the proof given below is totally elementary and different from the more analytic proof of Barth.

Theorem 1. *Let A be a simple abelian variety, and X and Y two closed irreducible subvarieties of A such that* $\dim X + \dim Y = \dim A$. *Then the intersection of X and Y is non-empty.*

For X and Y subvarieties of A, define $X - Y$ to be the subvariety $X - Y = \{x - y | x \in X, y \in Y\}$, and similarly define $X + Y$. With this notation, the above theorem is equivalent to saying that $0 \in X - Y$. But since we can translate X (or Y) by an arbitrary element of A without changing the hypothesis of the theorem (that $\dim X + \dim Y = \dim A$), we must prove that $X - Y = A$. Therefore Theorem 1 is equivalent to the following theorem.

Theorem 2. *Let A be a simple abelian variety, and X and Y two closed irreducible subvarieties of A such that* $\dim X + \dim Y = \dim A$. *Then* $X - Y = A$.

Notation. For a smooth point x on an algebraic variety X, we let $T_x(X)$ denote the tangent space to X at x. For an abelian variety A and points x and y on A, the tangent spaces $T_x(A)$ and $T_y(A)$ are canonically isomorphic. In particular, if $V \subset T_x(A)$ and $W \subset T_y(A)$ are subspaces, we can talk of $V + W$.

Proof of Theorem 2: Let

$$\begin{aligned} n : X \times Y &\rightarrow A \\ (x, y) &\rightarrow x - y. \end{aligned}$$

We have to prove that n is surjective. Let Z be the image of n. Z is a closed subvariety of A. Since any two open sets in $X \times Y$ intersect, there is a smooth point

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$(x_0, y_0) \in X \times Y$ whose image z_0 under n is a smooth point of Z. If $Z \neq A$, then $\dim Z < \dim A = \dim(X \times Y)$, and all the fibres of the map n will have positive dimensions. Let C be a closed curve in $X \times Y$ passing through (x_0, y_0) such that $n(C) = z_0$. Let C' be an open curve contained both in the smooth part of C and in the smooth part of $X \times Y$. Under the mapping n, the tangent spaces $T_x(X) + T_y(Y)$ for $(x, y) \in C'$ go to $T_{z_0}(Z)$ which is a proper subspace of $T_{z_0}(A)$. Let $\tilde{C}$ be the smooth part of the image of C' under the projection of $X \times Y$ onto X (or, onto Y if this image is a point). Therefore $\tilde{C}$ is curve in X such that $T_x(X) \subset T_{z_0}(Z)$ for all $x \in \tilde{C}$. In particular, $T_x(\tilde{C}) \subset T_{z_0}(Z)$ for all $x \in \tilde{C}$. Therefore the following lemma completes the proof.

Lemma 1. *There is no smooth curve $\tilde{C}$ in a simple abelian variety A such that $T_x(\tilde{C}) \subset V$ for all $x \in \tilde{C}$ for a fixed proper subspace V of the tangent space to A at the identity element.*

Proof. We assume without loss of generality that $0 \in \tilde{C}$, and let C be the closure of $\tilde{C}$. Let $C^{m,n}$ be the image of $C^{m+n} \to A$ under the mapping $(z_1, \ldots, z_m, w_1, \ldots, w_n) \mapsto z_1 + \cdots + z_m - (w_1 + \cdots + w_n)$. Similarly define $\tilde{C}^{m,n}$. The $C^{m,n}$ are closed irreducible subvarieties of A, and therefore there are natural numbers (m_0, n_0) such that $C^{k,l} \subset C^{m_0,n_0}$ for all $(k, l) \in \mathbf{N} \times \mathbf{N}$. Clearly this C^{m_0,n_0} is an abelian subvariety of A which, since A has no non-zero abelian subvarieties, must be A. It follows that $\tilde{C}^{m_0,n_0}$ contains an open subset of A. But on the other hand it is clear that the image of the differential of the mapping from $\tilde{C}^{m+n}$ to A at any point of $\tilde{C}^{m+n}$ is contained in V. Therefore the image $\tilde{C}^{m,n}$ can't contain an open set. The proof is therefore complete by contradiction.

As a simple corollary of Bézout's theorem we obtain the well known result (true in arbitrary characteristic) that there are no non-constant maps from a simple abelian variety to any variety of smaller dimension.

Remark. It has been pointed out by Dan Abromovich that this lemma is not true in positive characteristic and he offered the following counter example. Let $\pi : A \to B$ be a purely inseparable isogeny of simple abelian varieties in $\mathrm{ch}(p) > 0$ such that $d\pi \not\equiv 0$. Let C be a curve passing through the origin of A and having the origin as a smooth point and such that the tangent space to C at the origin is transversal to the kernel of $d\pi$. Then $\pi(C)$ will be a generically smooth curve such that the tangent space to its smooth points are contained in a fixed proper subspace of the tangent space of B at the identity.

We do not know if Theorems 1 and 2 are true in positive characteristic or not. As they are clearly true for the case of a curve and a hypersurface, the first case when we do not know whether these theorems are true is for two surfaces in a 4-dimensional simple abelian variety.

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