A Report on Artin's holomorphy conjecture

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February 23, 2007

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1 Introduction

The purpose of this paper is to present a report on the current status of Artin's holomorphy conjecture. For a fascinating account of how Artin was led to defining his L-series and his 'reciprocity law' see [19].

2 Preliminaries

2.1 Definition and properties of Artin's L-series

We begin with the definition of Artin's L-series. Let L/K be a finite normal extension of number fields with Galois group G. For a prime $\mathfrak{p}$ of K and $\mathfrak{q}$ a prime in L above $\mathfrak{p}$ let $G_{\mathfrak{q}}$ denote the decomposition subgroup and $I_{\mathfrak{q}}$ the inertia subgroup of G corresponding to $\mathfrak{q}$:

$$G_{\mathfrak{q}} = \{g \in G | g(\mathfrak{q}) = \mathfrak{q}\}; \quad I_{\mathfrak{q}} = \{g \in G_{\mathfrak{q}} | g(x) \equiv x \pmod{\mathfrak{q}}\}.$$

Let $\sigma_{\mathfrak{q}}$ denote the canonical generator, the Frobenius at $\mathfrak{q}$, of the cyclic group $G_{\mathfrak{q}}/I_{\mathfrak{q}}$. Note that if $\mathfrak{q}'$ is another prime in L above $\mathfrak{p}$ then the corresponding decomposition group, inertia group and Frobenius are conjugates in G of $G_{\mathfrak{q}}$, $I_{\mathfrak{q}}$ and $\sigma_{\mathfrak{q}}$. Let $\mathbf{N}$ denote the norm from K to $\mathbf{Q}$. If a group G acts on a vector space $\mathbf{V}$ and H is a subgroup of G then let $\mathbf{V}^H$ denote the subspace of H-fixed vectors of $\mathbf{V}$, i.e., $\mathbf{V}^H = \{v \in \mathbf{V} | h(v) = v, \forall h \in H\}$.

Definition: Suppose $\rho : G \longrightarrow \mathrm{GL}(\mathbf{V})$ is a representation of G on an n-dimensional complex vector space $\mathbf{V}$. Following Artin one can associate an L-series to ρ: for $\Re(s) > 1$ put

$$L(s,\rho) = \prod_{\mathfrak{p}} \frac{1}{\det(1 - \rho|_{\mathbf{V}^{I_{\mathfrak{q}}}}(\sigma_{\mathfrak{q}})\mathbf{N}\mathfrak{p}^{-s})},$$

the product ranging over all finite primes of K.

The product above converges for $\Re(s) > 1$ and hence defines an analytic function in that region. Since determinant remains invariant under conjugation, the local factors do not depend upon the choice of the prime $\mathfrak{q}$ in L above $\mathfrak{p}$. Further, since two isomorphic representations have the same determinant character, $L(s,\rho)$ depends only on the character χ of the representation ρ so that we may as well write $L(s,\chi)$ instead of $L(s,\rho)$. In fact, Artin gave the following explicit expression of $L(s,\rho)$ which is written purely in terms of the character of a representation:

$$\log L(s,\chi) = \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{\chi(\sigma_{\mathfrak{q}}^n)}{n \cdot \mathbf{N}\mathfrak{p}^{ns}} \quad \text{where} \quad \chi(\sigma_{\mathfrak{q}}^n) = \frac{1}{|I_{\mathfrak{q}}|} \sum_{\tau \in \sigma_{\mathfrak{q}}^n I_{\mathfrak{q}}} \chi(\tau).$$

Examples:

1. Suppose χ_0 is the trivial character of G. Then the Artin L-series $L(s,\chi_0)$ is nothing but the Dedekind zeta function $\zeta_K(s)$ of K.

2. Suppose χ_R is the regular representation of G. Then the Artin L-series $L(s,\chi_R)$ is the Dedekind zeta function $\zeta_L(s)$ of L. More generally, if H is a subgroup of G with M the subfield of L fixed by H and χ_{R_H} is the character of the left-regular representation of G on functions on G/H then $L(s,\chi_{R_H})$ is the Dedekind zeta function $\zeta_M(s)$ of M.

Properties:

1. If ρ_1, ρ_2 are two representations with characters χ_1, χ_2 then

$$L(s,\chi_1+\chi_2) = L(s,\chi_1)L(s,\chi_2).$$

   This enables one to define an Artin L-series for virtual characters of G. For example, (with $L/M/K$ as above) $\zeta_L(s)/\zeta_M(s)$ is the Artin L-series associated to the character $\chi_{R_H} - \chi_0$ of G.

2. If H is a subgroup of G and χ a character of H then

$$L(s,\operatorname{Ind}\chi) = L(s,\chi)$$

   where $\operatorname{Ind}\chi$ is the character of G induced from the character χ of H.

3. If H is a normal subgroup of G with $\lambda : G \longrightarrow G/H$, the canonical map and χ a character of the quotient group G/H then

$$L(s,\operatorname{Infl}\chi) = L(s,\chi)$$

   where $\operatorname{Infl}\chi$ is the character $\lambda \circ \chi$ of G .

2.2 Artin's conjecture

Conjecture 2.1 *If ρ does not contain the trivial representation then $L(s, \rho)$ has an analytic continuation as a holomorphic function to the whole of the complex plane.*

A consequence of this conjecture is that for any finite extensions of number fields M/K, $\zeta_K(s)$ *divides* $\zeta_M(s)$, i.e., $\zeta_M(s)/\zeta_K(s)$, which is the Artin L-series of the character $\chi_{R_H} - \chi_0$, should be a holomorphic function on $\mathbf{C}$. (Here L is a Galois extension of K containing M with Galois group G and H is the subgroup of G fixing M.)

This statement is known as *Dedekind's conjecture*. In fact, Artin seems to have been led to his L-series while trying to prove this statement which was proved for pure cubic fields by Dedekind in 1873. Artin himself was able to prove such a result when M/K is an Icosahedral extension (and also for some intermediate extensions therein), [1]. In the case of a normal extension L/K Dedekind's conjecture was proved without assuming Artin's conjecture by R. Brauer, [5] and independently by Aramata, (see 3.2 below). In the direction of Dedekind's conjecture in the non-normal case Uchida, [22], and Van der Waal, [23], (independently) have proved the following

Theorem 2.1 *(Uchida, Van der Waal): Let M/K be an extension of number fields and $\tilde{M}$ a normal closure of M. Suppose that Gal$(\tilde{M}/K)$ is solvable. Then $\zeta_M(s)/\zeta_K(s)$ is entire.*

3 Meromorphicity

3.1 Artin's reciprocity law

In [2] Artin had conjectured that if G is abelian then $L(s, \rho)$ is nothing but a L-series associated to certain characters introduced and studied by E. Hecke in 1917, [12]; Hecke had proved analytic continuation and functional equation for these abelian L-series.

Hecke characters are idele class characters $\chi : C_K \longrightarrow \mathbf{C}^*$ while Artin's L-series are associated to (characters of) representations of $\mathrm{Gal}(L/K)$. Thus, in order to identify his L-series with Hecke L-series for L/K abelian, Artin needed an identification of the abelian Galois group $\mathrm{Gal}(L/K)$ with a quotient of the idele class group such that for a prime $\mathfrak{p}$ of K and $\mathfrak{q}$ a prime in L

above $\mathfrak{p}$ the Frobenius at $\sigma_{\mathfrak{q}}$ corresponds to the uniformising parameter at $\mathfrak{p}$ in C_K. In 1927 Artin proved this 'reciprocity law' in [3] thereby proving his conjecture in the case of abelian G.

3.2 Representations of finite groups and meromorphicity

In 1930, [4] Artin proved the following general result about representations of finite groups.

Theorem 3.1 *(Artin): Every character of a finite group G is a linear combination with rational coefficients of induced characters from cyclic subgroups.*

This shows that an integral power of $L(s, \rho)$ is a product of Hecke L-series and hence is meromorphic. In the case of the regular representation of G, ρ_R, Brauer was able to prove the following, [5], pp 244.

Lemma 3.1 *(Brauer): The character $\chi_R - \chi_0$ of G can be expressed as a linear combination, with* positive *rational coefficients with denominator $|G|$, of characters of G which are induced by nontrivial irreducible characters of cyclic subgroups.*

This implies that some integral power of $\zeta_L(s)/\zeta_K(s)$, which is the Artin L-series for the character $\chi_R - \chi_0$ of G, is a product of abelian (Hecke) L-series corresponding to nontrivial characters. Since $\zeta_L(s)/\zeta_K(s)$ is meromorphic and Artin's conjecture is known to be true for abelian extensions we have the

Theorem 3.2 *(Aramata-Brauer): The quotient $\zeta_L(s)/\zeta_K(s)$ is entire.*

In [6] R. Brauer was able to improve on Artin's theorem; he proved the following theorem.

Theorem 3.3 *(Brauer): Every character of a finite group G is a linear combination with integer coefficients of characters induced from one dimensional characters of subgroups.*

This showed that $L(s,\rho)$ itself is meromorphic and proved Artin's conjecture for those representations which are *positive* integral linear combinations of one-dimensional representations, and hence also for those representations which are positive rational linear combinations of one-dimensional representations.

In [17] Stark proved the following result. The proof is so simple and elegant that we have decided to include it.

Theorem 3.4 *(Stark): If* $Ord_{s=s_0}(\zeta_L(s)) \leq 1$ *then for any character* χ *of the Galois group* G *of* L *over* K*,* $L(s,\chi)$ *is analytic at the point* $s = s_0$*.*

Proof : Let χ be an irreducible representation of the Galois group G of L over K, and ψ an irreducible representation of a subgroup H of G. (The subgroup H will be defined later.) Let n_χ and n_ψ denote the order of zeros (or poles, counted with negative multiplicity) of $L(s_0,\chi)$ and $L(s_0,\psi)$ respectively. Define generalised characters,

$$\begin{aligned} \theta_G &= \sum_\chi n_\chi \chi \\ \theta_H &= \sum_\psi n_\psi \psi. \end{aligned}$$

We will prove that θ_G is an irreducible character of dimension 1, and therefore $n_\chi \geq 0$, proving the theorem. For this purpose, let

$$\mathrm{Ind}_H^G \psi = \sum_\chi a(\psi,\chi)\chi.$$

Since $L(s,\psi) = L(s, \mathrm{Ind}_H^G \psi)$,

$$n_\psi = \sum_\chi a(\psi,\chi) n_\chi.$$

By Frobenius reciprocity,

$$\chi|_H = \sum_\psi a(\psi,\chi)\psi.$$

Therefore,

$$\theta_G|_H = \sum_{\chi,\psi} n_\chi a(\psi,\chi)\psi = \sum n_\psi \psi = \theta_H \qquad (*).$$

Now let $g \in G$, and H the cyclic subgroup of G generated by g. Since H is an abelian group, Artin's conjecture is true for representations of H. Factorisation of the zeta function of L as the product of the L-functions attached to characters of H and the fact that $Ord_{s_0}(\zeta_L(s)) \leq 1$ implies that $\theta_H = \psi$ for some linear character of H. From the equation $(*)$, this means that $\theta_G(g)$ is a root of unity for all $g \in G$. By Schur orthogonality,

$$1 = \frac{1}{|G|} \sum |\theta_G(g)|^2 = \sum n_\chi^2.$$

Therefore, exactly one n_χ is non-zero. Now take $H = \{1\}$ in $(*)$. This gives $\theta_G(1) = 1$, implying $n_\chi \cdot \chi(1) = 1$, which implies $n_\chi = 1$ and $\chi(1) = 1$.

Using the ideas in [17], Foote and Kumar Murty, [11], were able to reprove Artin's conjecture for some characters, already covered by Brauer's theorem, *without* writing the characters as positive linear combinations of monomial characters. They do this by comparing the order of zero or pole of the $L(s, \chi)$ at a fixed point, as χ varies over the irreducible characters of G, with the order of the Dedekind zeta function $\zeta_L(s)$ at the same point. They also proved the following

Theorem 3.5 *(Foote-Kumar Murty): Let L/K be a Galois extension of number fields with soluble Galois group G and let $p_1 < p_2 < \cdots < p_n$ be the distinct prime divisors of $|G|$. If $\zeta_L(s)$ has a zero of order r at $s = s_0$, where $r \leq p_2 - 2$, then $L(s, \chi)$ is analytic at s_0 for all irreducible characters χ of G.*

In [14], Michler has explicitly described the relation between the orders of zeros of Dedekind zeta functions and the orders of zeros or poles of Artin L-functions at a point in $\mathbf{C} - \{1\}$ when the Galois group of the extension is S_n, the symmetric group of degree n. Suppose $\mathrm{Gal}(L/K) = S_n$. Denote the set of all partitions λ of n by $\mathcal{P}(n)$. The irreducible characters χ_μ of S_n are parametrised by the partitions $\mu \in \mathcal{P}(n)$. For each $\lambda \in \mathcal{P}(n)$ there is a unique Young subgroup Y_λ of S_n; let $L_\lambda = L^{Y_\lambda}$ be its fixed field. The main result of [14] asserts that for each point s_0 in $\mathbf{C} - \{1\}$ and each partition $\mu \in \mathcal{P}(n)$, the order $n_\mu(s_0)$ of zero or pole of the Artin L-series $L(s, \chi_\mu)$ is uniquely determined by the orders of zero $r_\lambda(s_0)$ of the Dedekind zeta functions $\zeta_{L_\lambda}(s)$ of the fixed fields L_λ for some $\lambda \in \P(n)$.

3.3 Functional equation

It follows from Brauer's theorem (theorem 3.3) that Artin L-functions can be written as a quotient of product of Hecke L-functions. Since Hecke L-functions have meromorphic continuation to all of the complex plane, and a functional equation, so do Artin L-functions. In this section we state the functional equation satisfied by an Artin L-function. Before we can do that, we must recall the definition of Artin conductor and gamma factors associated to representations of the Galois group.

For every representation V of the Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/K)$, there is an integral ideal of K, denoted by $f(V)$ and called the Artin conductor of the representation V, which is divisible by exactly those primes in K which are ramified in the representation V. The Artin conductor has the property that $f(V_1 \oplus V_2) = f(V_1)f(V_2)$ for any two representations V_1 and V_2 of $\mathrm{Gal}(\bar{\mathbf{Q}}/K)$. For one dimensional representations of $\mathrm{Gal}(\bar{\mathbf{Q}}/K)$, the Artin conductor is the conductor of the corresponding Grossencharacter. There is a formula for Artin conductor involving the higher ramification groups for which we refer to the book of Tate [20]. Here we just mention that if the representation V of $\mathrm{Gal}(\bar{\mathbf{Q}}/K)$ is induced from a representation χ of $\mathrm{Gal}(\bar{\mathbf{Q}}/M)$, where M is an extension of K, then one has

$$f(V) = D(M/K)^{\chi(1)} Norm(f(\chi))$$

where $D(M/K)$ is the discriminant of M over K, and $f(\chi)$ is the Artin conductor of the character χ.

Suppose that the representation V of $\mathrm{Gal}(\bar{\mathbf{Q}}/K)$ factors through the Galois group of a field extension L of K. Let w be a place of L over an Archimedian place v of K. This defines an element in G of order 1 or 2 which in the representation V has let's say n_v^+, $+1$ eigenspaces, and n_v^-, -1 eigenspaces. If v is a real place, define a gamma factor

$$L_v(s, V) = [\pi^{-s/2}\Gamma(s/2)]^{n_v^+}[\pi^{-(s+1)/2}\Gamma((s+1)/2)]^{n_v^-}.$$

If v is a complex place, define $L_v(s, V) = [2\cdot(2\pi)^{-s}\Gamma(s)]^n$. With this notation, the *completed Artin L-function*

$$\Lambda(s, \chi) = \{|d_K|^{\chi(1)} N_{K/\mathbf{Q}}(f(\chi))\}^{s/2} \prod_{v|\infty} L_v(s, \chi) \cdot L(s, \chi)$$

satisfies the functional equation:

$$\Lambda(1 - s, \chi) = W(\chi)\Lambda(s, \overline{\chi})$$

where $W(\chi)$ is a non-zero complex number of absolute value 1 called the *Artin root-number.*

4 Two-dimensional representations

4.1 Dihedral, Tetrahedral and Octahedral Representations

If ρ is a 2-dimensional Galois representation over a number field k then ρ determines a faithful representation of $\mathrm{Gal}(K/k)$ where K is the finite normal extension of k corresponding to the kernel of the representation and therefore ρ corresponds to a finite subgroup of $GL(2, \mathbf{C})$. The finite subgroups of $GL(2, \mathbf{C})$ have been classified by F. Klein: their image in $PGL(2, \mathbf{C})$ is one of the following.

1. Cyclic.
2. Dihedral.
3. Tetrahedral (isomorphic to A_4).
4. Octahedral (isomorphic to S_4).
5. Icosahedral (isomorphic to A_5).

Thus any finite subgroup of $GL(2, \mathbf{C})$ is a cyclic central extension of one of the above projective groups. The inverse images of the subgroups A_4, S_4, A_5 of $SO(3, \mathbf{R})$ in $SU(2)$ which is the 2 fold cover of $SO(3, \mathbf{R})$:

$$0 \to \mathbf{Z}/2 \to SU(2) \to SO(3, \mathbf{R}) \to 1,$$

are $\tilde{A}_4 = SL_2(\mathbf{Z}/3)$, $\tilde{S}_4 = GL_2(\mathbf{Z}/3)$, and $\tilde{A}_5 = SL_2(\mathbf{Z}/5)$. The group $\tilde{A}_4$ has representations of dimensions 1,1,1,3,2,2,2, of which the last 3 are non-trivial on $\mathbf{Z}/2$. The group $\tilde{S}_4$ has representations of dimensions 1,1,3,3,2,2,2,4, of which the last 3 are non-trivial on $\mathbf{Z}/2$. The group $\tilde{A}_5$ has representations of dimensions 1,3,3,5,4,2,2,4,6, of which the last 4 are non-trivial on $\mathbf{Z}/2$.

Artin's conjecture in the cyclic and dihedral case follows from Artin reciprocity and Hecke's proof that abelian L-series are entire (section 3.1). Langlands proved Artin's conjecture for tetrahedral and Tunnell for octahedral representations. Thus the only case where Artin's conjecture is not known over $\mathbf{Q}$ is the case of the icosahedral representations (section 5).

Langlands has infact generalised Artin's conjecture to ask whether the L-function arising from an irreducible n-dimensional Galois representation of a number field k is infact the L-function of a cusp form on $GL(n, k)$, because then by known properties of the L-function of automorphic representations of $GL(n, k)$, it will in particular follow that an Artin L-function is entire if $n > 1$. This generalisation of Artin's conjecture which asks if an Artin L-function is the L-function of an automorphic L-function is called *strong Artin conjecture*. Using the theory of base change, Langlands was able to prove such a statement for 2 dimensional tetrahedral representations, which was then extended to Octahedral case by Tunnell.

Theorem 4.1 *(Langlands-Tunnell): A two dimensional representation of* $\mathrm{Gal}(\overline{\mathbf{Q}}/k)$ *with values inside a finite solvable subgroup of* $GL(2, \mathbf{C})$ *comes from an automorphic representation of* $GL(2, \mathbf{A}_k)$*.*

It follows from the 'converse' theory developed by Weil and Langlands that Artin's conjecture for 2 dimensional Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ implies that there is a natural one-to-one correspondence between equivalence classes of odd two-dimensional Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and cusp forms of weight 1 in such a way that the L-series associated (by Artin) to a Galois representation agrees with the L-series associated (by Hecke) to the corresponding cusp form. This conjectural correspondence is to be viewed as a natural generalisation of the equivalence, coming from class field theory, of characters of the Galois group (of an abelian extension) and Hecke characters (Artin's reciprocity). These correspondences are special cases of the Langlands' programme, see [15]. Here is the theorem of Weil and Langlands.

Theorem 4.2 *(Langlands-Weil): Let* ρ *be an odd 2-dimensional Galois representation over* $\mathbf{Q}$ *with conductor* N *and determinant* ϵ*. If* $L(s, \rho \otimes \lambda)$ *is an entire function for all twists* $\rho \otimes \lambda$ *of* ρ *by one-dimensional representations* λ*, then there is a cusp form* f *of weight 1 on* $\Gamma_1(N)$ *with nebentypus character* ϵ *such that* $L_f(s) = L(s, \rho)$*.*

The following theorem due to Deligne and Serre constructs Galois representations associated to cusp forms of weight 1.

Theorem 4.3 *(Deligne-Serre): If* f *is a new form of type* $(1, \epsilon, N)$ *then there is an odd two-dimensional Galois representation* $\rho : Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{C})$ *with conductor* N *and determinant* ϵ *such that* $L_f(s) = L(s, \rho)$*.*

4.2 Two dimensional representations of prime conductor

In this section which has been completely taken from the paper of Serre [16], we summarise the classification of two dimensional representations ρ of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ of prime conductor. We will say that ρ is dihedral, or A_4, S_4, A_5 depending on the image of ρ in $PGL(2, \mathbf{C})$. We will denote the determinant of ρ by ϵ.

Dihedral case: Two dimensional dihedral representations ρ of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ of prime conductor p arises only for $p \equiv 3 \bmod 4$ and is then induced from a non-trivial unramified character of $K = \mathbf{Q}(\sqrt{-p})$. Such Galois representations indeed correspond to a cusp form on $\Gamma_1(p)$ as is well-known from Hecke theory.

Non-dihedral case : Suppose that ρ is a two dimensional non-dihedral representation of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ of prime conductor p. Then,

(1) $p \not\equiv 1 \bmod 8$.

(2) If $p \equiv 5 \bmod 8$, ρ is of type S_4, and the character ϵ of order 4 and conductor p.

(3) If $p \equiv 3 \bmod 4$, ρ is of type S_4 or A_5, and ϵ is the Legendre symbol $n \to \left(\frac{n}{p}\right)$.

Conversely, start with a Galois extension L of $\mathbf{Q}$ and a prime number p. Consider the following 3 cases.

(i) $\text{Gal}(L/\mathbf{Q}) \cong S_4$ and $p \equiv 5 \bmod 8$.

(ii) $\text{Gal}(L/\mathbf{Q}) \cong S_4$ and $p \equiv 3 \bmod 4$.

(iii) $\text{Gal}(L/\mathbf{Q}) \cong A_5$ and $p \equiv 3 \bmod 4$.

An embedding of $\text{Gal}(L/\mathbf{Q})$ in $PGL(2, \mathbf{C})$ determines a projective representation $\tilde{\rho}_L$ of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. There exists a lifting of $\tilde{\rho}_L$ to $GL(2, \mathbf{C})$ with prime conductor p, if and only if one has the following in the 3 respective cases above:

(i) L is the normal closure of a non-real quartic field of discriminant p^3.

(ii) L is the normal closure of a quartic field of discriminant $-p$.

(iii) L is the normal closure of a non-real quintic field of discriminant p^2.

4.3 Tate's example

Till 1976 Artin's conjecture was known to be true only for representations which are *positive* rational linear combinations of one-dimensional representations in which cases some integral power of the corresponding Artin L-functions are product of Hecke L-functions. In 1976, in [19] Tate reported on the construction of a tetrahedral representation of conductor $N = 133$ thereby inagurating the experimental verification of Artin's conjecture. They were able to prove the existence of the corresponding new-form of weight 1 and level 133 predicted by the Langlands' conjecture ("by relatively easy hand computation"). By the theorem of Deligne-Serre (theorem 4.3 above), this produced the first example of an Artin L-series which is known to be holomorphic in spite of the fact that no power of it is a product of abelian L-series.

The strategy used by Tate is the following: Given a 2-dimensional representation ρ of the appropriate kind, compute the coefficients a_n of the Artin L-series $L(s, \rho)$ for $n \leq A$, say. Then look for a modular form of weight 1 of with Fourier coefficients a_n for $n \leq A$. If A is sufficiently large, for instance $A \geq (N/12) \prod_{p|N}(1 + p^{-1})$, then this form is uniquely determined, if it exists. Now invoke the theorem of Deligne-Serre to get a representation ρ_1 corresponding to this form for which the Artin's conjecture is true.

4.4 Dimension of spaces of new forms of weight 1

Let $S_1^{\mathrm{new}}(N, \epsilon)$ denote the complex vector space of new forms of weight 1, level N and character ϵ, and let $d(N, \epsilon)$ be the number of equivalence classes of 2-dimensional, complex, irreducible, continuous, odd Galois representations with conductor N and determinant ϵ. The theorem of Deligne-Serre implies $\dim S_1^{\mathrm{new}}(N, \epsilon) \leq d(N, \epsilon)$. Artin's conjecture is the statement that

$$\dim S_1^{\mathrm{new}}(N, \epsilon) = d(N, \epsilon).$$

Thus one way to verify Artin's conjecture in a particular case is by computing the numbers $\dim S_1^{\mathrm{new}}(N, \epsilon)$ and $d(N, \epsilon)$. But there is no known method of computing the dimension of the space of forms of weight 1. Also, there is no general way of determining the number of equivalence classes of 2-dimensional, complex, irreducible, continuous, odd Galois representations with conductor N and determinant ϵ.

Suppose $N = q$, a prime with $q \equiv 3 \pmod 4$. Hecke proved that

$$f_\chi = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) e^{2\pi i \mathbf{N}(\mathfrak{a}) z}$$

where χ is an arbitrary non-trivial character on the ideal class group of $\mathbf{Q}(\sqrt{-q})$ and $\mathfrak{a}$ runs over all non-zero integral ideals in $\mathbf{Q}(\sqrt{-q})$, is a cusp form of weight 1, level q and character $\left(\frac{\cdot}{q}\right)$. Since $q \equiv 3 \bmod 4$, the class number of $\mathbf{Q}(\sqrt{-q})$ is odd. It is easy to see that $f_\chi = f_{\chi'}$ if and only if $\chi = \chi'$, or $\chi = \chi'^{-1}$. It follows that if the class number of $\mathbf{Q}(\sqrt{-q})$ is h, then there are atleast $(h-1)/2$ linearly independent cusp forms of weight 1, level q and character $\left(\frac{\cdot}{q}\right)$. Therefore Siegel's theorem estimating the order of the class group implies the ineffective lower bound

$$\mathrm{dim} S_1(q, \left(\frac{\cdot}{q}\right)) >>_\epsilon q^{1/2-\epsilon}$$

for all $\epsilon > 0$. But there are also forms which are not of the Hecke type. However, forms of non-Hecke type are rare and it is conjectured that

$$\mathrm{dim} S_1(q, \left(\frac{\cdot}{q}\right)) = \frac{1}{2}(h-1) + \mathrm{O}_\epsilon(q^\epsilon).$$

In [9] W. Duke has proved that

$$\mathrm{dim} S_1(q, \left(\frac{\cdot}{q}\right)) << q^{11/12} \log^4 q,$$

with an absolute implied constant.

5 Icosahedral representations

This section is devoted to the last unsolved case of the Artin's conjecture for 2-dimensional representations, the case of the icosahedral representation. Following the method of Tate, J. Buhler in 1978 constructed for the first time a 2-dimensional icosahedral representation satisfying Artin's conjecture, [7] (see 5.1). About 15 years later, in 1993 Frey and his students discovered seven more examples, [10]. All these examples are for icosahedral representations over $\mathbf{Q}$.

5.1 Verification of Artin's conjecture

The aim of this section is to outline the method followed by J. Buhler [7] who gave the first example of an icosahedral representation for which Artin's conjecture is true. The construction of these examples involve extensive computation on a computer.

The starting point is the construction of icosahedral representations with low conductors; the lowest conductor found was $800 = 2^5 5^2$. This was done by sieving through a large number of quintic polynomials, eliminating first those whose discriminants were not squares, then reducible polynomials and then eliminating those whose Galois groups were proper subgroups of A_5. For each surviving quintic the ring of integers is computed to determine the behavior of ramified primes and the minimal conductor. The quintic which gives rise to the icosahedral representation of conductor 800 is:

$$F(x) = x^5 + 10x^3 - 10x^2 + 35x - 18.$$

Denoting by ρ the icosahedral representation of conductor 800 associated to $F(x)$, the next step is the calculation of the $L(s, \rho) = \sum a_n n^{-s}$. This calculation involves computations in a sextic extension of $\mathbf{Q}$. As a sample we reproduce a few values of a_n; note that a_n is 0 if n is not prime to 10. Let i denote a fixed fourth root of 1 and j the positive root of $x^2 - x - 1 = 0$.

n	$\rightarrow$	1	3	7	9	11	13	17	19
a_n	$\rightarrow$	1	$-i$	$-ij$	-1	0	j	0	$i - ij$

A cusp form of weight 1 is said to be dihedral if the corresponding representation (via Deligne-Serre) is an irreducible properly induced representation and that the form is tetrahedral (resp. octahedral, icosahedral) if the representation is tetrahedral (resp. octahedral, icosahedral).

Consider the space V of modular forms of type $(1, \epsilon, 800)$ (i.e., of weight 1, character ϵ and level dividing 800) for a carefully chosen Dirichlet character ϵ. The choice of ϵ is such that in V:

- there are two non-cuspidal eigen forms of level 100, denoted by g_1, g_2; and
- there is only one dihedral form of level 100, g_3 (note that g_3 is a cusp form).

Each of these forms can 'pushed-up' to level 800 by the Atkin-Lehner operators, $B_d,\ d = 1, 2, 4, 8$ $(g|B_d(z) = g(dz))$. Let

$$g_{i,d} = g_i|B_d.$$

If g is a modular form of type $(1, \epsilon, N)$ let $\bar{g}$ denote the 'complex conjugate' of g; $\bar{g}$ is of type $(1, \bar{\epsilon}, N)$ and the Fourier coefficients of $\bar{g}$ are the complex conjugates of the Fourier coefficients of g.

For z in the upper half plane, let $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$ where a_n are as above $(L(s, \rho) = \sum a_n n^{-s})$. The aim is to show that this f coincides with the q-expansion of a cusp form of weight 1 up to a required number of terms. This is done as follows.

By searching in the space of cusp forms of weight 2 and level 800 (which has dimension 97) Buhler discovered the

Fact: For each $i = 1, 2, 3$ and $d = 1, 2, 4, 8$ there is a modular form $h_{i,d}$ of weight 2 and level 800 (and trivial character) such that

$$f\bar{g}_{i,d} \equiv h_{i,d} \pmod{q^{360}}.$$

Here $\pmod{q^M}$ means that the first M terms of the two power series agree.

¿From this it follows that

$$h_{i,d}\bar{g}_{j,d'} \equiv h_{j,d'}\bar{g}_{i,d}$$

for all i, j, d, d'. On each side of this congruence is a modular form of weight 3, level 800 and a certain character. Such modular forms are sections of a bundle on $X_0(800)$ whose degree is 344. Therefore the above congruence is actually an equality and we have that

$$f' = \frac{h_{i,d}}{\bar{g}_{i,d}}$$

is independent of the choice of i, d. It turns out that f' is a cusp form of type $(1, \epsilon, 800)$. It is to prove that it is so that one uses the representation of f' in so many ways as above by showing that the forms $g_{i,d}$ have no common zero in the upper half-plane.

That f' is not of the dihedral is proved by looking at the eigenvalue of f' for the Hecke operator T_3; and that it is not of the tetrahedral or octahedral type follows from:

- There are no cyclic extensions of $\mathbf{Q}$ of degree 3 unramified outside 2 and 5 and hence there are no A_4 extensions of $\mathbf{Q}$ unramified outside 2 and 5.

- There are exactly three S_4 extensions of $\mathbf{Q}$ unramified outside 2 and 5 and the corresponding representations have conductors not dividing 800.

Therefore it follows that there is an icosahedral form in V! If ρ_1 denotes the corresponding representation (via Deligne-Serre) then ρ_1 is an icosahedral representation which satisfies the Artin's conjecture. It is not known if ρ_1 is same as the representation ρ that we started with. Buhler ([7], pp 90) proves that if f' is an eigenform for T_{11} as well then it is so.

5.2 Recent developments

Like the Deligne-Serre theorem which was stated as theorem 4.3 which constructs Galois representations associated to modular forms of weight 1 such that the L-function associated to the Galois representation and the modular form is the same, there is a more classical theorem due to Deligne which constructs ℓ-adic Galois representations for modular forms of any weight $k \geq 2$. These Galois representations can be reduced modulo the maximal ideal in an appropriate ring of ℓ-adic integers to construct Galois representations with values in $GL(2)$ of a finite field. There has been much activity in recent years motivated by a conjecture of Serre which says that any irreducible Galois representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with values in a finite field which is irreducible and is *odd* in the sense that the determinant of complex conjugation is -1, comes via this construction for some modular form.

A representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with values in $GL(2, \mathbf{C})$ in fact lies in $GL(2, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers of a number field. By going modulo prime ideals in $\mathcal{O}$, an Artin representation therefore gives rise to representations of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with values in $GL(2)$ of a finite field. Kevin Buzzard and Richard Taylor have in [8] proved that if this representation is modular in the sense that it comes from a modular form via Deligne's construction, then under a mild hypothesis, Artin's conjecture is true for the original representation.

We note the following result, cf. theorem 1.2 in [18], which relates elliptic curves to $\bmod 5$ Galois representations from which the modularity of certain representations of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with values in $GL_2(\mathbf{Z}/5)$ follows from the work of Wiles, Taylor-Wiles, and Diamond which proves modularity of certain elliptic curves over $\mathbf{Q}$.

Theorem 5.1 *If $\bar{\rho} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{Z}/5)$ is any representation whose determinant is the cyclotomic character mod 5, then there exists an elliptic curve over $\mathbf{Q}$ which realises $\bar{\rho}$ on its 5 torsion points.*

However, for the purposes of Artin's conjecture, modularity of a representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with values in $GL_2(\mathbf{Z}/5)$ is not adequate as it is easy to see that the reduction modulo a prime ideal of an Artin representation of A_5-type taking values in $GL_2(\mathcal{O})$ cannot lie in $GL_2(\mathbf{Z}/5)$ (with determinant the cyclotomic character).

In conclusion, one can say that there is now a general approach to Artin's conjecture for 2 dimensional representations of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ which in the absence of knowing that enough mod$p$ representations of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ are modular, has not yet been successful in completing Artin's conjecture for 2 dimensional representations of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$.

In a recent work, C. Khare has also proved that Serre's conjecture implies Artin's conjecture for 2 dimensional odd representations of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, cf. [13].

6 Stark's Conjecture

In this section we state an important conjecture of Stark which does not depend on Artin's conjecture but uses Artin L-functions. There is a lot of importance attached to Stark's conjecture especially as it is a contribution to Hilbert's 12th problem about generation of classfields of number fields in terms of explicit values of transcendental functions just as was done for $\mathbf{Q}$ and quadratic imaginary fields by attaching values of exponential and elliptic functions respectively.

If L is a number field with r_1 real embeddings and r_2 pairs of complex embeddings, then it is known that the zeta function of L has a zero of order $r_1 + r_2 - 1$ at the origin, and the leading term of the Taylor expansion of $\zeta_L(s)$ at $s = 0$ begins as

$$\zeta_L(s) = -\frac{hR}{e} \cdot s^{r_1+r_2-1} + \cdots,$$

where h is the class number of L, R the regulator, e the number of roots of unity in L.

Stark has defined an anologue of the regulator of a number field for all Artin representations, now called Stark regulator and has conjectured that just like the Dedekind zeta function, the leading term of the Taylor expansion of $L(s,\chi)$ at $s = 0$ divided by the Stark regulator is an algebraic number beloging to the cyclotomic field in which the values of the character of χ lies. We refer to the book of Tate [20] for an exposition of Stark's conjecture which remains open. We note that from proposition 3.4 of [20], the order of vanishing of $L(s,\chi)$ at $s = 0$ is $r(\chi)$ which is

$$r(\chi) = \sum_{v\in\infty}(\dim V^{G_w}) - \dim V^G,$$

where V is the vector space on which the representation χ is defined, and G_w denotes the decomposition group at a place w of L over an infinite place v of K. It follows from this formula for $r(\chi)$ that if L is abelian over K and χ is a faithful character, then $r(\chi) = 1$ if and only if the decomposition group is non-trivial at exactly one infinite place of K. In this case, the Stark regulator is the determinant of a 1×1 matrix, and unwinding the definitions, it follows from Stark's conjecture that if L is an abelian extension of a number field K, and χ is a character of the Galois group of L over K, then

$$L'(0,\chi) = -\frac{1}{e}\sum_{\sigma}\chi(\sigma)\log|\epsilon^\sigma|,$$

for a suitable element ϵ in L, where e denotes the number of roots of unity in L; the absolute value is taken with respect to a fixed place of L which lies over the unique Archimedean place of K where the decomposition group is non-trivial. The element ϵ of L is called the Stark unit and has the property that the extension of L obtained by attaching the e-th root of ϵ is abelian over K. If χ is a faithful character of $\mathrm{Gal}(L/K)$, ϵ generates L over K. This case of Stark's conjecture dealing with abelian extensions is also open except when $K = \mathbf{Q}$, or an imaginary quadratic field. The case $K = \mathbf{Q}$ being well-known, and the imaginary quadratic case due to Stark where the Stark unit is an *elliptic unit*.

Acknowledgement: The authors would like to thank C. Khare for some helpful remarks.

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