SELF-DUAL REPRESENTATIONS OF DIVISION ALGEBRAS AND WEIL GROUPS: A CONTRAST

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Abstract. Irreducible selfdual representations of any group fall into two classes: those which carry a symmetric bilinear form, and the others which carry an alternating bilinear form. The Langlands correspondence, which matches the irreducible representations σ of the Weil group of a local field k of dimension n with the irreducible representations π of the invertible elements of a division algebra D over k of index n, takes selfdual representations to selfdual representations. In this paper we use global methods to study how the Langlands correspondence behaves relative to this distinction among selfdual representations. We prove in particular that for n even, σ is symplectic if and only if π is orthogonal. More generally, we treat the case of $\mathrm{GL}_m(B)$, for B a division algebra over k of index r, and $n = mr$.

Introduction. Let ρ be a selfdual representation of a group G on a vector space V over $\mathbb{C}$. We will say that ρ is *orthogonal*, resp. *symplectic*, if G leaves a nondegenerate *symmetric*, resp. *alternating*, bilinear form $B : V \times V \to \mathbb{C}$ invariant. If ρ is irreducible, exactly one of these possibilities will occur, and we may define a *sign*, also called *parity*, $c(\rho) \in \{\pm 1\}$, taking it to be $+1$ in the orthogonal case and -1 in the symplectic case.

Now let k be a local field of characteristic 0. The groups of interest to us will be on the one hand, $G = \mathrm{GL}_m(D)$, where D is a division algebra with center k and index d, and on the other hand, the Weil group W_k.

The local Langlands correspondence [HT, Hen1] when used in conjunction with the Jacquet-Langlands correspondence [Bad], gives a bijection $\pi \to \sigma$, satisfying certain natural properties, between the discrete series representations of G and the set of equivalence classes of irreducible representations σ of W_k' of dimension $n = md$. Here W_k' denotes W_k if k is archimedean, and the extended Weil group $W_k \times \mathrm{SL}_2(\mathbb{C})$ if k is non-archimedean. One calls σ the *Langlands parameter* of π. It is immediate from the construction that π is selfdual if and only if σ is, cf. [HT, page 2, property 5]. However, it is not *a priori* clear whether the local Langlands reciprocity respects the sign, i.e., whether $c(\pi)$ should equal $c(\sigma)$. The main result of this paper is the following.

THEOREM A. *Let $n = md$, D a division algebra of index d over a local field k of characteristic zero, $G = \mathrm{GL}_m(D)$, and π an irreducible selfdual discrete series*

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representation of G with parameter σ—an irreducible representation of W'_k of dimension n. Then we have

$$(-1)^m c(\pi) = (-1)^n c(\sigma)^m.$$

COROLLARY B. *Let π be an irreducible, selfdual representation of $D^\times$, for any division algebra D of index n over a local field k of characteristic zero, and let σ be the Langlands parameter of π. If n is odd, π is always orthogonal, while if n is even,*

$$\pi \text{ orthogonal} \Longleftrightarrow \sigma \text{ symplectic}.$$

Indeed when n is odd, σ is necessarily orthogonal, and so Theorem A implies, for any factorization $n = md$ and $G = \mathrm{GL}_m(D)$ with D a division algebra of index d, that $c(\pi) = +1$, i.e., π is orthogonal. For $m = 1$, we get $c(\pi) = (-1)^{d+1}c(\sigma)$, which implies that an irreducible selfdual representation π of $D^\times$, D of even index d, is symplectic if and only if σ is orthogonal. This surprising *flip* is what we noticed first for $d = 2$, spurring our interest in the general case, which is more subtle to establish. Based on considerations of Poincaré duality on the middle dimensional cohomology of certain coverings of the Drinfeld upper-half space, we conjectured in [PR] the assertion of Corollary B, and established some positive results in [PR, Pra], including the case of $n = 2$. In [Pra] it was proved that if d is odd and if the residual characteristic of k is odd, then $D^\times$ has no selfdual irreducible representations of dimension > 1, showing that in this case, the conjecture is difficult only in the even residual characteristic. A program to prove Corollary B (for $G = D^\times$) along the geometric lines, using cohomological methods involving the formal moduli of Lubin-Tate groups, has been announced in the supercuspidal case by Laurent Fargues; it does not seem, however, that, without further input, his suggested methods would work for general discrete series representations, nor for $\mathrm{GL}_m(D)$ with $m > 1$.

Corollary B associates, to each irreducible, symplectic Galois representation σ of dimension n (even), a *new secondary invariant*, defined by whether or not the associated orthogonal representation π of $D^\times$, D of index n, lifts to the (s)pin group. This aspect was investigated for $n = 2$ in [PR].

Our proof of Theorem A for non-archimedean k proceeds by using *global methods*, made possible by the following *product formula*.

THEOREM C. *Let F be a global field, $G = \mathrm{GL}_m(\mathbb{D})$, where $\mathbb{D}$ is a division algebra over F and Z the center of G. Suppose $\Pi = \otimes'_v \Pi_v$ is an irreducible, selfdual discrete automorphic representation of $G(\mathbb{A}_F)$ of central character ω. Then we have*

$$\prod_{v \in \mathrm{ram}(\mathbb{D})} c(\Pi_v) = 1,$$

where $\mathrm{ram}(\mathbb{D})$ denotes the set of places where $\mathbb{D}$ is ramified.

As a consequence, we see that in the case $m = 1$ and $\mathrm{ram}(\mathbb{D}) = \{u, v\}$, we have $c(\Pi_u) = c(\Pi_v)$, i.e., Π_u and Π_v have the same parity. In particular, if we know one, we know the other. Thanks to this product formula, given an irreducible selfdual representation π of $D^\times$, with D a division algebra over a local field k, our strategy is to find a number field F with $F_v = k$ for a place v of F, a division algebra $\mathbb{D}$ over F with $\mathrm{ram}(\mathbb{D}) = \{u, v\}$, and a *selfdual* automorphic representation Π of $\mathbb{D}^\times(\mathbb{A}_F)$ such that $\Pi_v = \pi$ and Π_u a representation for which Theorem A can be checked. Thanks to the Jacquet-Langlands correspondence between $\mathbb{D}^\times$ and GL_n, we see that it suffices to find a *selfdual* discrete automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_F)$ with Π'_v, resp. Π'_u, being associated to Π_v, resp. Π_u. The difficulty is not so much in globalizing per se, but in choosing a global Π' which is also *selfdual*. This has been made possible by the work of Jiang and Soudry in the appendix to this paper.

In Section 3 we check Theorem A for representations of $D^\times$ trivial on $D^\times(1)$, which can be used at the place u, allowing a foothold on all the representations.

We would like to mention that at the moment the globalization of local selfdual representations of GL_n (of the same sign) to a global selfdual representation is available only for supercuspidal representations, and not yet for discrete series representations; in particular, we are not allowed to prescribe Steinberg representations at some local places, which would have simplified many considerations in the paper. This creates problems for those representations of $D^\times$ whose Jacquet-Langlands lift to GL_n is not supercuspidal, which are then dealt with using Mœglin-Waldspurger's description in [MW] of residual spectrum of GL_n, on which the general Jacquet-Langlands correspondence has been established by Badulescu [Bad]. This strategy works with only minor modifications for $\mathrm{GL}_m(D)$ too.

One of the subtle points (see Section 5) of our proof is that as we go back and forth between the local and global correspondences, we might not come back to the same local representation π which we started with, but rather to the representation $i(\pi)$ obtained from π by applying the Aubert-Zelevinsky involution i. Luckily, π and $i(\pi)$ turn out to have the same sign.

The statement of Theorem A for n odd is the simplest as it does not refer to the Langlands parameter at all: $c(\pi) = 1$ for any irreducible selfdual discrete series representation of $\mathrm{GL}_m(D)$ with $n = md$ odd. It says in particular that a selfdual irreducible representation of $D^\times$ for a division algebra of index d, an odd integer, must be orthogonal. Since $D^\times/k^\times$ is a profinite group, the question is clearly in the realm of finite group theory. However, our proof uses many recent and nontrivial results in the theory of Automorphic representations to achieve this. The recent work of Bushnell and Henniart has given a local proof of this result for $D^\times$ in [BH2].

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of proof, allowing us to embed a local supercuspidal representation of $\mathrm{GL}(n)$ of orthogonal or symplectic type into a global automorphic representation of the same type.

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1. Results we need. In this section we will review two key results we will need, the first concerning globalization, due to Jiang and Soudry, described in the appendix, and the second concerning the functorial correspondence between the discrete automorphic forms of inner forms of $\mathrm{GL}(n)$, due to Badulescu.

THEOREM 1.1. *Let K be a number field, and v_i, $i = 1, \ldots, d$ finite places of K. Let K_{v_i} be the corresponding local fields. Suppose that the π_i are irreducible selfdual supercuspidal representations of $\mathrm{GL}_n(K_{v_i})$ whose parameters τ_i are either all orthogonal, or all symplectic. Then there exists a selfdual cuspidal automorphic representation Π on $\mathrm{GL}_n(\mathbb{A}_K)$ with π_i as the local component of Π at each of the places v_i.*

We will explain why this follows from the results in the appendix. Suppose π is an irreducible, selfdual supercuspidal representation of $\mathrm{GL}_n(k)$ with parameter τ, where k is a non-archimedean local field with residue field $\mathbb{F}_q$. Then, by the local Langlands correspondence, τ is an irreducible n-dimensional representation of W_k, the Weil group. For any representation (σ, V) of W_k, its L-function is defined by $L(s,\sigma) = \det(I - q^{-s}(\varphi|V^I))$, with φ being the Frobenius and I the inertia group. It follows that $L(s,\sigma)$ has a pole at $s = 0$ if and only if the trivial representation of W_k is contained in σ. Taking σ to be $r(\tau)$, where r is either the exterior square, or the symmetric square, representation of $\mathrm{GL}_n(\mathbb{C})$, one sees (the well known fact) that τ is symplectic or orthogonal if and only if $L(s, r(\tau))$ has a pole at $s = 0$ (for the appropriate r). Moreover, since τ is irreducible, it is a representation of $\mathrm{Gal}(\overline{k}/k)$ up to twisting by an unramified character ν, and the selfduality of τ forces ν to be of finite order, which may be chosen to be 1. Then, being a continuous $\mathbb{C}$-representation, τ factors through a finite Galois group, resulting in all the inverse roots of φ being of absolute value 1. Thus $L(s, r(\tau))$ has no pole at $s = 1$. Now let $L^{\mathrm{Sh}}(s, \pi, r)$ denote the L-factor attached to π and

$r \in \{\Lambda^2, \mathrm{Sym}^2\}$ by Shahidi. Then by a theorem of Henniart [Hen3], $L(s, r(\tau))$ and $L^{\mathrm{Sh}}(s, \pi, r)$ coincide. (The local Langlands reciprocity only gives the equality of $L(s, \Lambda^2(\tau))L(s, \mathrm{Sym}^2(\tau))$ and $L^{\mathrm{Sh}}(s, \pi, \Lambda^2)L^{\mathrm{Sh}}(s, \pi, \mathrm{Sym}^2)$.) In the work of Jiang and Soudry described in the appendix, they use the L-factor $L(s, \pi, r)$ coming from the Rankin-Selberg theory involving an integral representation. Their gamma factor $\gamma(s, \pi, r)$ associated to (π, r), which is, up to an invertible holomorphic function, the quotient of $L(s, \pi, r)$ by $L(1-s, \pi, r)$ (as π is selfdual), coincides with the corresponding Shahidi gamma factor. Then any pole of $L(s, r(\tau))$ at $s = 0$ results in a corresponding one of $L(s, \pi, r)$ at the same point, unless this factor has a pole at $s = 1$ as well. But if $L(s, \pi, r)$ has a pole at $s = 1$, then $L(s, \pi \times \pi)$, which is known to equal the Shahidi L-factor $L^{\mathrm{Sh}}(s, \pi \times \pi)$, will also have a pole there, contradicting the fact that $L(s, r(\tau))$ has no pole at $s = 1$. Thus $L(s, \pi, r)$ has a pole whenever τ is symplectic, resp. orthogonal, for $r = \Lambda^2$, resp. Sym^2. Theorem 1.1 is now implied by Theorem A4 (of the appendix).

Next, the following form of the Jacquet-Langlands correspondence between $\mathrm{GL}(n)$ and its inner forms, due to Badulescu, cf. [Bad, Theorem 5.1], will also play crucial role for us in the proof of Theorem A; see also [HT, Theorem VI.1.1]. We will denote by i the Aubert-Zelevinsky involution (cf. [Bad, Section 2.6], for example).

THEOREM 1.2. (Badulescu-Jacquet-Langlands) *Let Π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ of unitary central character $\omega : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ which occurs in the discrete part of $L^2(G(K)Z(\mathbb{A}_K)\backslash G(\mathbb{A}_K), \omega)$. Suppose $\mathbb{B}$ is a central simple algebra over K of dimension n^2. Let S be the finite set of places where $\mathbb{B}$ is not split. Assume S has only non-archimedean places. Then Π can be transferred to a discrete automorphic representation Π' on $\mathbb{B}^\times(\mathbb{A}_K)$ if at every place v in S, Π_v is either a discrete series representation or is a Speh representation. Moreover, either Π_v, or its Aubert-Zelevinsky involution $i(\Pi_v)$, has the same parameter as Π'_v. Finally, any automorphic representation of $\mathbb{B}^\times(\mathbb{A}_K)$ appears with multiplicity 1 in the discrete spectrum.*

Here by a Speh representation we mean the unique irreducible quotient representation, for a parabolic P associated to a partition $(m, m, \ldots, m)$ of $n = rm$, of the representation of $\mathrm{GL}_n(K_v)$ induced from P by $(\rho \otimes |\det|^{(r-1)/2}) \times (\rho \otimes |\det|^{(r-3)/2}) \times \cdots \times (\rho \otimes |\det|^{(1-r)/2})$, with ρ a supercuspidal representation of $\mathrm{GL}_m(K_v)$.

2. Orthogonality and reality. The following basic lemma is presumably well known, but for lack of an appropriate reference outside the realm of compact groups, we supply a proof.

LEMMA 2.1. *An irreducible admissible, unitary representation (π, V) of a p-adic group G carries a nonzero symmetric bilinear form $B : V \times V \to \mathbb{C}$ if and* *only if π is defined over $\mathbb{R}$, i.e., there is a G-invariant real subspace W of V such that $V = W \otimes_{\mathbb{R}} \mathbb{C}$.*

Proof. Recall that a complex vector space V is defined over $\mathbb{R}$ if and only if there is a complex conjugation on V, i.e., an involution $v \to \bar{v}$ on V such that $\overline{zv} = \bar{z}\bar{v}$ where z belongs to $\mathbb{C}$.

Let $H : V \times V \to \mathbb{C}$ be a positive definite G-invariant Hermitian form on V. We will prove that if V is defined over $\mathbb{R}$ (and therefore has a complex conjugation: $v \to \bar{v}$), then V carries a nonzero G-invariant symmetric bilinear form. Define,

$$B(v,w) = H(v,\bar{w}).$$

Clearly, $B(\lambda v, \mu w) = \lambda\mu B(v,w)$ for $\lambda, \mu \in \mathbb{C}$; further B is a G-invariant nonzero bilinear form. It remains to check that B is symmetric, which amounts to the relation

$$H(v,w) = \overline{H(\bar{v},\bar{w})},$$

for $v, w \in V$.

In any case, by the uniqueness (up to a positive real number) of a positive-definite Hermitian form on V, there exists a $\lambda > 0$ such that

$$H(v,w) = \lambda\overline{H(\bar{v},\bar{w})}.$$

Since $\bar{\bar{v}} = v$, $\lambda^2 = 1$, which means that $\lambda = 1$ as it is positive, proving that B is symmetric.

Conversely, we show that if there exists a symmetric G-invariant bilinear form $B : V \times V \to \mathbb{C}$, then V is defined over $\mathbb{R}$, by constructing an involution $v \to \bar{v}$ on V (which is G-invariant, and conjugate linear).

This part of the proof will use the fact that since V is admissible, any smooth linear form on V, i.e., a linear form which is left invariant by a compact open subgroup of G, is of the form $v \to H(v,w)$ for a unique w in V. This allows us to define a map, $w \to w'$ on V by

$$B(v,w) = H(v,w'),$$

for all $v, w \in V$. Clearly, $w \to w'$ is conjugate linear. Using the fact that B is symmetric, it is easy to see that $w \to w'$ is an involution if and only if

$$H(v,w) = \overline{H(v',w')}.$$

However, this is not true in general. In any case, by uniqueness (up to a positive real number) of positive definite Hermitian forms,

$$H(v,w) = \lambda\overline{H(v',w')},$$

for some $\lambda > 0$. It is then easy to check that $v \to \bar{v} = \sqrt{\lambda} v'$ is an involution, giving rise to a real structure on V, proving the lemma. □

COROLLARY 2.2. *Let (π, V) be an irreducible unitary representation of a p-adic group G, and H a closed subgroup of G. Assume that the space of linear forms $\ell : V \to \mathbb{C}$ which are H-invariant is one-dimensional. Then if π is selfdual, it is orthogonal.*

Proof. It suffices to prove that the representation (π, V) of G can be defined over $\mathbb{R}$. This is equivalent to proving that there is a conjugate linear automorphism φ of V commuting with the G-action with $\varphi^2 = 1$. Since V is unitary and selfdual, V is isomorphic to $\bar{V}$, and therefore there is a conjugate linear automorphism ϕ of V commuting with the G-action which by Schur's lemma has $\phi^2 = \mu$ for some μ in $\mathbb{C}^\times$. We will prove that ϕ can be scaled to achieve $\mu = 1$.

Because of the uniqueness of the H-invariant linear form ℓ,

$$\overline{\ell(v)} = \lambda \ell(\phi(v)),$$

for all $v \in V$ (for some $\lambda \in \mathbb{C}^\times$). Applying this identity to $\phi(v)$ instead of v, and noting that $\phi^2(v) = \mu \cdot v$, we find that

$$\bar{\lambda}^{-1} \ell(v) = \lambda \mu \ell(v).$$

Therefore $\mu^{-1} = \lambda \bar{\lambda}$, and changing ϕ to $\varphi = \lambda \phi$, we have $\varphi^2 = 1$, proving the corollary. □

COROLLARY 2.3. *Let π_1 be an irreducible unitary selfdual representation of a p-adic group G, H a closed subgroup of G, and π_2 an irreducible unitary selfdual representation of H such that* $\mathrm{Hom}_H(\pi_1, \pi_2) \cong \mathbb{C}$. *Then π_1 and π_2 have the same parity.*

Proof. The previous corollary applied to the representation $\pi_1 \boxtimes \pi_2^\vee$ of the group $G \times H$ containing the subgroup $\Delta(H) \hookrightarrow H \times H \hookrightarrow G \times H$ proves this assertion. □

Now consider the Aubert-Zelevinsky involution $\pi_1 \to i(\pi_1)$ (cf. [Bad, Section 2.6]) defined on the Grothendieck group of smooth representations of a p-adic reductive group G as an alternating sum of parabolically induced representations of the various Jacquet modules of π_1. The involution $\pi_1 \to i(\pi_1)$ is known to send an irreducible representation to another irreducible representation $|i(\pi_1)|$ equal to $i(\pi_1)$ up to a sign.

PROPOSITION 2.4. *Let G be a reductive algebraic group over a non-archimedean local field k. Then an irreducible representation π is orthogonal if and only if $|i(\pi)|$ is orthogonal.*

Proof. The assertion follows by combining Lemma 2.1 with the fact that both induction and the Jacquet functor take real representations to real representations; cf. the lemma below. □

LEMMA 2.5. *Let K be subfield of $\mathbb{C}$, and G a reductive p-adic group. Let $R_K(G)$ denote the Grothendieck group of admissible representations of G over K which are of finite length when tensored with $\mathbb{C}$. Then an irreducible representation π of G (over $\mathbb{C}$) which belongs to $R_K(G)$ can be defined over K.*

Proof. We recall the proof of this well-known result for finite groups, which works in the p-adic context too. The essential points of the proof are:

(1) The set of isomorphism classes of irreducible representations of G over K form a basis of $R_K(G)$.

(2) For two distinct irreducible representations π_1 and π_2 of G over K (of finite length over $\mathbb{C}$), the representations $\pi_1 \otimes_K \mathbb{C}$ and $\pi_2 \otimes_K \mathbb{C}$ are semi-simple and have no irreducible representations in common.

We will leave the proofs of these assertions to the reader, but observe that it clearly proves the lemma by writing $\pi = V_1 - V_2$ in the Grothendieck group of representations of G over K, with V_1, V_2 representations of G over K, and writing each V_i as a sum of irreducible representations of G over K. □

3. The product formula: proof of Theorem C. Preserving the notations of Theorem C, define a $G(\mathbb{A}_F)$-invariant bilinear form B on Π by

$$(f,g) \longrightarrow \int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} fg\,d\mu, \quad \forall f,g \in \Pi,$$

where $d\mu$ is an invariant measure on $G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. We check that this is a non-degenerate bilinear form on Π. Note that the space of functions spanned by $\bar{f}$ (the complex conjugate of f), as f varies over Π, gives rise to the representation $\Pi^\vee$, which is isomorphic to Π. Hence by the multiplicity 1, $\bar{f} \in \Pi$. Since

$$B(f,\bar{f}) = \int f\bar{f}\,d\mu \neq 0,$$

the bilinear form B on Π is non-degenerate. It is evidently symmetric too, proving that $c(\Pi) = 1$.

Now $c(\Pi_v) = 1$ for all places v where the division algebra D is unramified is a consequence of the following result, proving Theorem C.

PROPOSITION 3.1. *Every irreducible admissible selfdual representation π of $\mathrm{GL}_n(k)$ over any local field k is orthogonal.*

Proof. First let k be *non-archimedean*. Let $U(k)$ be the subgroup of $\mathrm{GL}_n(k)$ consisting of upper-triangular unipotent matrices. By a theorem of Zelevinsky [Zel, Corollary to Theorem 8.1], for any representation π of $\mathrm{GL}_n(k)$, there is a character

$\psi : U(k) \to \mathbb{C}^\times$ which appears in π as a quotient with multiplicity 1. We may assume that π is not one-dimensional, which forces ψ to be non-trivial. Since ψ is not selfdual, we cannot directly apply Corollary 2.3, but a small modification works. Let s be the diagonal element $(1, -1, 1, \ldots, (-1)^{n-1})$, which normalizes $U(k)$ and takes the character ψ to its inverse. Put $H := \langle s \rangle \ltimes U(k) \subset \mathrm{GL}_n(k)$. Then the 2-dimensional representation τ of H induced by the character ψ of $U(k)$ is an irreducible orthogonal representation; cf. Proposition 4.1 below. Moreover, by Frobenius reciprocity, $\mathrm{Hom}_H(\pi, \tau)$ is $\mathrm{Hom}_{U(k)}(\pi, \psi) = \mathbb{C}$. So Corollary 2.3 applies, with $\pi_1 = \pi$ and $\pi_2 = \tau$, to yield the desired conclusion.

It is left to prove the Proposition when k is *archimedean*.

It is a general result due to D. Vogan [Vog, page 2, Theorem 1.2] that any irreducible $(\mathfrak{g}, K)$-module has a minimal K-type which occurs with multiplicity 1. This completes the proof for $\mathrm{GL}_n(\mathbb{R})$, since for its maximal compact subgroup $\mathrm{O}_n(\mathbb{R})$, every irreducible representation carries an invariant symmetric bilinear form by the lemma below. For $\mathrm{GL}_n(\mathbb{C})$, the minimal K-type is unique—the authors owe this remark to D. Vogan—as a general fact for any Complex Lie group due to Zelobenko, see [Žel2], and therefore a selfdual representation of $\mathrm{GL}_n(\mathbb{C})$ has a selfdual minimal K-type. By the following well-known lemma, every irreducible, selfdual representation of the compact groups $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{U}_n(\mathbb{R})$ carries an invariant symmetric bilinear form, concluding the proof of this lemma. (Note the difference between $\mathrm{U}_n(\mathbb{R})$ and $\mathrm{SU}_n(\mathbb{R})$; there are already irreducible symplectic representations of $\mathrm{SU}_2(\mathbb{R})$, but they do not extend to selfdual representations of $\mathrm{U}_2(\mathbb{R})$.) □

LEMMA 3.2. *Every irreducible, selfdual representation of* $\mathrm{U}_n(\mathbb{R})$ *is orthogonal. Every irreducible representation of* $\mathrm{O}_n(\mathbb{R})$ *is selfdual and orthogonal.*

Proof. We supply one of the many possible proofs for the convenience of the reader. For a compact connected Lie group G with maximal torus T, and w_0 the longest element in the Weyl group of T (with respect to some ordering of positive roots), the dual of a finite dimensional irreducible representation π_λ with highest weight λ is $\pi_{-w_0(\lambda)}$. For $\mathrm{U}_n(\mathbb{R})$, let T be the diagonal torus, and $w_0(t_1, t_2, \ldots, t_n) = (t_n, \ldots, t_2, t_1)$. Therefore highest weights of irreducible selfdual representations are of the form:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq -\lambda_2 \geq -\lambda_1.$$

By the well-known branching law from $\mathrm{U}_n(\mathbb{R})$ to $\mathrm{U}_{n-1}(\mathbb{R})$, cf. [Žel1, Section 132, pp. 385–387], we find that an irreducible selfdual representation of $\mathrm{U}_n(\mathbb{R})$ contains an irreducible selfdual representation of $\mathrm{U}_{n-1}(\mathbb{R})$ with multiplicity one; therefore by induction on n, every irreducible selfdual representation of $\mathrm{U}_n(\mathbb{R})$ is orthogonal.

For $\mathrm{SO}_{2n+1}(\mathbb{R})$, and $\mathrm{SO}_{4n}(\mathbb{R})$, $w_0 = -1$, so every irreducible representation of these groups is selfdual. For $\mathrm{SO}_{2n}(\mathbb{R})$, with n odd, the inner conjugation action of

$O_{2n}(\mathbb{R})$ takes a representation of $SO_{2n}(\mathbb{R})$ to its dual, hence every irreducible representation of $O_{2n}(\mathbb{R})$ is selfdual for n odd also. By the (multiplicity freeness of) branching from $O_n(\mathbb{R})$ to $O_{n-1}(\mathbb{R})$, cf. [Žel1, Section 129, Corollaries of Thmeorems 2 and 3, pp. 378–379], we are done again. $\square$

This completes the proof of Theorem C.

Having considered the case of $GL_n(\mathbb{R})$, and $GL_n(\mathbb{C})$, it is natural to consider the case of $GL_n(\mathbb{H})$, for $\mathbb{H}$ the unique quaternion division algebra over $\mathbb{R}$, specially since this case also can be handled along similar lines; however, we will not have any occasion to use this result in this paper.

PROPOSITION 3.3. *An irreducible admissible selfdual representation* π *of* $GL_n(\mathbb{H})$ *is orthogonal if and only if its central character is trivial.*

Proof. The proof follows from the existence of a minimal K-type, appearing with multiplicity one, for K a maximal compact subgroup of $GL_n(\mathbb{H})$, which in this case is the compact form of $Sp_n(\mathbb{C})$ all whose irreducible representations are selfdual, and an irreducible representation of $Sp_n(\mathbb{C})$ is orthogonal if and only if its central character is trivial. $\square$

4. Sign in the Level 1 case. In this section we prove Theorem A about irreducible selfdual representations of $D^\times/D^\times(1)$ where $\{D^\times(i)\}$, $i = 1,2,\ldots$, denotes the standard decreasing filtration, with $D^\times(i)$ being the subgroup of $D^\times$ consisting of elements in $\mathcal{O}_D^\times$ congruent to the identity modulo ϖ_D^i. Here $\mathcal{O}_D$ denotes a maximal order of D and ϖ_D the uniformizing parameter. The global proofs in this paper involve a reduction step using the product formula, and this approach depends crucially, in most situations, on this local input, which represents the simplest of the situations for Theorem A.

Various aspects of the representation theory of $D^\times/D^\times(1)$ are analyzed in the work of Silberger and Zink in [SZ]. We begin with some notation, and recalling the parametrization of the irreducible representations of $D^\times/D^\times(1)$ from [SZ] (called level zero representations there).

As in the rest of the paper, let D be a division algebra with center a non-archimedean local field k, and of index n. Suppose that $n = ef$, and let k_f be the unramified extension of k of degree f, contained in D. Let D_f be the centralizer of k_f in D which is a division algebra with center k_f and of index e. A character χ of $k_f^\times$ will be called *regular* if all its Galois conjugates are distinct. For a character χ of $k_f^\times$, let $\tilde{\chi}$ be the character of $D_f^\times$ obtained by composing χ with the reduced norm mapping $\mathrm{Nrd} : D_f^\times \to k_f^\times$. If the character χ is tame, i.e., trivial on $k_f^\times(1)$, then $D^\times(1)$ normalizes $(D_f^\times, \tilde{\chi})$, and hence the character $\tilde{\chi}$ of $D_f^\times$ can be extended to a character of $D^\times(1)D_f^\times$ by declaring it to be trivial on $D^\times(1)$, which, by abuse of notation, we again denote by $\tilde{\chi}$.

With this notation, it follows from Clifford theory as in [SZ] that the dimensions of irreducible representations of $D^\times/D^\times(1)$ are divisors of n, and that there is a bijection between irreducible representations of $D^\times/D^\times(1)$ of dimension f and $\mathrm{Gal}(k_f/k)$-orbits of regular characters of $k_f^\times$ which are trivial on $k_f^\times(1)$ obtained by inducing the character $\tilde{\chi}$ of $D^\times(1)D_f^\times$ to $D^\times$.

We analyze these representations to see if they are orthogonal or symplectic in the following general proposition.

PROPOSITION 4.1. *Let N be a normal subgroup of a group G of index $f>1$, such that G/N is a cyclic group of order f. Let ϖ be an element of G whose image in G/N is a generator of the cyclic group G/N. Let π be an irreducible representation of G of dimension f whose restriction to N contains a character $\chi: N \to \mathbb{C}^\times$, so that $\pi = \mathrm{Ind}_N^G \chi$. If χ is of order 2, then π is an orthogonal representation of G. Assume now that χ is not of order 2, then the following hold:*

(1) *If π is selfdual, then f is even, say $f=2d$.*

(2) *The representation π is selfdual (of dimension $f=2d$) if and only if $\chi^{-1} = \chi^{<d>}$, where $\chi^{<d>}(n) = \chi(\varpi^d n \varpi^{-d})$ for $n \in N$.*

(3) *If π is selfdual, then $\chi(\varpi^f) = \pm 1$, and $\chi(\varpi^f)=1$ if and only if π is an orthogonal representation.*

(4) *If π is selfdual, then it is orthogonal if and only if* $\det \pi(\varpi) = -1$.

Proof. If the character χ is of order 2, then clearly $\pi = \mathrm{Ind}_N^G \chi$ is an orthogonal representation. We will therefore in the rest of the proof assume that χ is not of order 2.

Since π is irreducible, all the conjugates $\chi^{<i>}$ of χ, defined by $\chi^{<i>}(n) = \chi(\varpi^i n \varpi^{-i})$ are distinct for $i = 0, 1, \ldots, f-1$. The representation $\pi = \mathrm{Ind}_N^G \chi$ is selfdual if and only if $\mathrm{Ind}_N^G \chi = \mathrm{Ind}_N^G \chi^{-1}$, i.e., if and only if $\chi^{-1} = \chi^{<i>}$ for some i. This implies that $2i \equiv 0 \bmod f$, which means that f must be even, say $f=2d$, and $i=d$.

As the element ϖ^f commutes with ϖ, $\chi^{<i>}(\varpi^f) = \chi(\varpi^f)$ for all i. Therefore ϖ^f operates on π by a scalar, which, if π is selfdual, must be ± 1.

Let e_0 be any nonzero vector in the space of π on which N operates via the character χ. Define $e_i = \pi(\varpi^i)e_0$ for $0 \le i \le f-1$. If $\pi(\varpi^f)e_0 = -e_0$, then

$$e_0 \wedge e_d + e_1 \wedge e_{d+1} + \cdots + e_{d-1} \wedge e_{2d-1},$$

is left invariant under both ϖ and N, hence the representation is symplectic. On the other hand, if $\varpi^f \cdot e_0 = e_0$, then the vector

$$e_0 \cdot e_d + e_1 \cdot e_{d+1} + \cdots + e_{d-1} \cdot e_{2d-1}$$

in $\mathrm{Sym}^2(\pi)$ is left invariant under both ϖ and N, hence π is orthogonal.

Since $\pi(\varpi)e_i = e_{i+1}$ for $i \in \mathbb{Z}/f$, if and only if π is orthogonal, it implies that if π is selfdual, it is orthogonal if and only if $\det \pi(\varpi) = -1$. □

Continuing now with the representations of $D^\times/D^\times(1)$ of dimension f with $ef = n$, let ϖ be an element of $D^\times$ which normalizes k_n, an unramified extension of k inside D of degree n over k, such that $\varpi^n = \varpi_k$, a uniformizer in k. The element ϖ of $D^\times$ projects to a generator of the cyclic group $D^\times/D^\times(1)D_f^\times \cong \mathbb{Z}/f\mathbb{Z}$, and as ϖ^f centralizes k_f, it lies in D_f. Since $(\varpi^f)^e = \varpi_k$, it follows that the reduced norm of ϖ^f is $(-1)^{e-1}\varpi_k$. From the previous proposition, we conclude the following corollary. (We note that a regular character of $k_f^\times$ which is trivial on $k_f^\times(1)$ cannot be of order 2.)

COROLLARY 4.2. *Let χ be a regular tame character of $k_f^\times$, and π_χ the associated representation of $D^\times/D^\times(1)$ of dimension f. Then π_χ is selfdual if and only if f is even, say $f = 2d$, and the character χ restricted to $k_d^\times$ is trivial on the index 2 subgroup consisting of norms from $k_f^\times$. Assuming π_χ to be selfdual, it is orthogonal if and only if χ restricted to $k_d^\times$ is trivial.*

Recall that the Weil group $W_{k_f/k}$ sits in the exact sequence,

$$1 \longrightarrow k_f^\times \longrightarrow W_{k_f/k} \longrightarrow \mathrm{Gal}\left(k_f/k\right) \longrightarrow 1.$$

As there is an element ϖ in $W_{k_f/k}$ which maps to the generator of the Galois group of k_f over k, and whose f-th power is a uniformizer in k, we have the following corollary for representations of the Weil group.

COROLLARY 4.3. *For a regular character μ of $k_f^\times$, let σ_μ be the induced representation of $W_{k_f/k}$ of dimension f. Then σ_μ is selfdual if and only if f is even, say $f = 2d$, and the character μ restricted to $k_d^\times$ is trivial on the index 2 subgroup consisting of norms from $k_f^\times$. Assuming σ_μ to be selfdual, it is orthogonal if and only if μ restricted to $k_d^\times$ is trivial, or if and only if* $\det\sigma_\mu$ *is nontrivial.*

The following result, correcting a mistake in the remark on page 182 of the paper of Silberger and Zink [SZ], was proposed by us to G. Henniart, and has since been proved by Bushnell and Henniart; see [BH1]. This completes the proof of theorem A for representations of $D^\times$ which are trivial on $D^\times(1)$. In this theorem, and in what follows, we will let sp_e denote the e-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$.

THEOREM 4.4. *The Langlands parameter of the representation π_χ of dimension f of $D^\times$ is $\sigma_\mu \otimes sp_e$, with σ_μ the f-dimensional representation of W_k induced by the character*

$$\mu := \chi\omega_2^{e(f-1)} : k_f^\times \longrightarrow \mathbb{C}^\times,$$

where ω_2 is the quadratic unramified character of $k_f^\times$.

Remark 4.5. In order to be able to make use of representations of level 1, we must know that there are irreducible orthogonal as well as symplectic representations of $D^\times/D^\times(1)$ of dimension f for any *even* divisor $f = 2d$ of n. By corollary 4.2, this reduces to a question over finite fields. A character of $\mathbb{F}_{q^{2d}}^\times$ extended to a tame character of $k_f^\times$ with values ± 1 on the uniformizer gives rise to an irreducible selfdual representation of $D^\times$ of dimension f if and only if it arises from the group of norm 1 elements of $\mathbb{F}_{q^{2d}}^\times$, denoted $\mathbb{S}^1(\mathbb{F}_{q^d})$, through the map $x \to x/\bar{x}$ for $x \in \mathbb{F}_{q^{2d}}^\times$, but does not factor through a smaller field under the norm mapping. Since $\mathbb{S}^1(\mathbb{F}_{q^d})$ is a cyclic group of order $(q^d + 1)$, one can consider characters on it of order $(q^d + 1)$. Such characters of $\mathbb{F}_{q^{2d}}^\times$ will not arise from an intermediate field through the norm mapping as those have orders divisible by $(q^s - 1)$, where s is a divisor of $2d$.

Remark 4.6. It may be noted that in case the index of D is odd but residue characteristic of k is 2, then although there are selfdual representations of $D^\times$, there are none of $D^\times/D^\times(1)$ which are not one dimensional. This follows since index being odd, we can twist a selfdual representation of $D^\times$ to assume that its central character is trivial, and by noting that $D^\times/(k^\times \cdot D^\times(1))$ is a group of odd order.

5. Proof of Theorem A. We now prove Theorem A in the following equivalent form.

THEOREM 5.1. *Let $n \geq 1$, $n = md$, k a non-archimedean local field, D a division algebra over k of index d, and π' an irreducible selfdual discrete series representation of* $\mathrm{GL}_m(D)$. *Then π' is orthogonal if d is odd. If d is even, and m is odd, then π' is orthogonal if and only if its parameter σ is symplectic. If both m and d are even, then the representation π' is orthogonal.*

Proof. Since π' is a discrete series representation of $\mathrm{GL}_m(D)$, its parameter σ is an irreducible representation of $W_k' = W_k \times \mathrm{SL}_2(\mathbb{C})$; write $\sigma = \tau \otimes sp_b$, with $ab = n$ and τ an irreducible, selfdual a-dimensional representation of W_k. The proof divides naturally into three cases, according to whether $b = 1$, or $b > 1$ with a is even, or $b > 1$ with a odd. Denote by π the irreducible selfdual representation of $\mathrm{GL}_n(k)$ corresponding to π' by the local Jacquet-Langlands correspondence.

Case 1 ($\mathbf{b = 1}$). In this case σ is trivial on $\mathrm{SL}_2(\mathbb{C})$, and so π is supercuspidal.

Suppose d *is odd*. As π is selfdual and supercuspidal, we may globalize it, by applying Theorem 1.1 (of Jiang-Soudry), to a selfdual, cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_K)$ with π as its local component at $K_v = k$. Let E be a cyclic extension of K of degree d such that v splits completely in E. Let Π_E denote the base change of Π to E. Let $\mathbb{B}$ be the central division algebra over E of dimension d^2 over E such that $\mathbb{B} \otimes_K K_v \cong D^d$, and such that $\mathbb{B}$ has no other ramification. The existence and uniqueness of such a division algebra $\mathbb{B}$ follows from classfield theory.

Let $\Pi_E^{\mathbb{B}}$ denote the automorphic representation of $\mathrm{GL}_m(\mathbb{B}(\mathbb{A}_E))$ obtained from Π_E by the global Jacquet-Langlands correspondence [Bad]. The component of $\Pi_E^{\mathbb{B}}$ at $E_v := E \otimes_K K_v \simeq k^d$ is isomorphic to ${\pi'}^{\otimes d}$. At every place u of E not lying over v, the u-component of $\Pi_E^{\mathbb{B}}$ is a representation of $\mathrm{GL}_n(E_u)$. Applying Proposition 3.1 and the product formula given by Theorem C, we find that the d-th power of $c(\pi')$ is trivial, and therefore π' is an orthogonal representation (as d is odd).

We are left to consider when d *is even*. Write the index as $d = 2r$, and the Brauer invariant of D as $\frac{s}{2r} \in \mathbb{Q}/\mathbb{Z}$ with $(s, 2r) = 1$. Let $\mathbb{D}$ be a division algebra over the number field K of index $2mr$ such that $\mathbb{D}$ gives rise to $\mathrm{M}_m(D)$ at one place v, and with Brauer invariants $\frac{-1}{2mr} \in \mathbb{Q}/\mathbb{Z}$ at sm other places, call them $u_1, \ldots, u_{sm}$. We may further take $\mathbb{D}$ to be split at all the remaining places. The existence of such a global division algebra follows from classfield theory.

Globalize π, using Theorem 1.1, to an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_K)$ such that its local components at the places $u_1, \ldots, u_{sm}$ correspond to representations of $\mathbb{D}_{u_i}^{\times}$ trivial on $\mathbb{D}_{u_i}^{\times}(1)$, and selfdual of the same parity as π (parity to be understood on the Galois side). Transporting this automorphic representation to $\mathbb{D}^{\times}(\mathbb{A}_K)$, and using the product formula furnished by Theorem C, we get

$$c(\pi) = \big[-c(\sigma)\big]^{sm} = \big[-c(\sigma)\big]^{m};$$

the first equality is because of our calculation of signs in the last section for selfdual representations of $\mathbb{D}_{u_i}^{\times}$ trivial on $\mathbb{D}_{u_i}^{\times}(1)$, and the second equality due to the fact that s is odd because of the condition earlier $(s, 2r) = 1$. This is equivalent to the conclusion of the theorem, therefore proving it when π is supercuspidal.

Case 2 ($\mathbf{b} \neq \mathbf{1}$ and $\mathbf{a}$ even). Let Σ be a selfdual cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A}_K)$ whose local component at the place v of K with completion k has Langlands parameter τ. We may assume, thanks to Theorem 1.1, that at some other finite places, say $u_1, u_2, \ldots, u_{sm}$, the local components Σ_{u_i} are supercuspidal of level 1, with parameters τ_{u_i} of the same parity as τ. It is at this stage that we need the restriction that τ is even dimensional as we are able to construct selfdual representations of $D^{\times}/D^{\times}(1)$ with parameter $\tau \otimes sp_b$ only for τ even dimensional. If the residue characteristic of k is odd, by [Pra, Proposition 4] any irreducible selfdual representation of the Galois group of k of dimension > 1 is even dimensional. Hence τ must be even dimensional unless it is a character of order 2.

By the work of Mœglin and Waldspurger [MW], $\Sigma = \otimes_v \Sigma_v$ gives rise to a selfdual representation in the residual spectrum of $\mathrm{GL}_n(\mathbb{A}_K)$ denoted by $\Sigma[b]$; the automorphic representation $\Sigma[b]$ is at each place u of K the unique irreducible quotient of the parabolically induced representation $\Sigma_u|\det|^{(b-1)/2} \times \cdots \times \Sigma_u|\det|^{-(b-1)/2}$ of $\mathrm{GL}_n(K_u)$. By Theorem 1.2 concerning the global Jacquet-Langlands correspondence due to Badulescu, the representation $\Sigma[b]$ of $\mathrm{GL}_n(\mathbb{A}_K)$ can be transported to an automorphic representation $\Sigma'[b]$ of $\mathbb{D}^{\times}(\mathbb{A}_K)$, where $\mathbb{D}$ is the division algebra of index n over K constructed earlier. Transporting the automorphic representation

$\Sigma[b]$ to $\mathbb{D}^\times(\mathbb{A}_K)$, and using the product formula (Theorem C), we get

$$c(\pi) = \big[-c(\sigma)\big]^{sm} = \big[-c(\sigma)\big]^{m},$$

the last equality due to the fact that s is odd because of the condition $(s, 2r) = 1$. This is equivalent to the conclusion of the theorem.

Case 3 ($\mathbf{b} \neq \mathbf{1}$ and $\mathbf{a}$ odd). If b is also odd, then D has odd index d because $ab = n = md$, and the proof given above works again, in which we globalize σ to an automorphic self-dual representation of $\mathrm{GL}_a(\mathbb{A}_K)$, and then use [MW] as above to construct $\Sigma[b]$, a selfdual representation in the residual spectrum of $\mathrm{GL}_n(\mathbb{A}_K)$. As the index of D is odd, we are able to conclude, as before, that $c(\pi) = c(\pi)^d = 1$.

It remains to deal with parameters of the form $\sigma = \tau \otimes sp_b$, $n = ab$, with τ an irreducible, selfdual a-dimensional representation of W_k with **a** *odd and* **b** *even*. In this case, we need to prove that the representation π' of $\mathrm{GL}_m(D)$ is orthogonal. Note that $md = n = ab$.

We will begin by proving the assertion assuming $\mathbf{m} = \mathbf{1}$, so that π' is a (self-dual) representation of $D^\times$, with the Brauer invariant of D being $\frac{s}{ab} \in \mathbb{Q}/\mathbb{Z}$ with $(s, ab) = 1$.

As τ is an irreducible selfdual representation of W_k of dimension a, it corresponds to a supercuspidal representation π_τ, say, of $\mathrm{GL}_a(k)$. We may globalize π_τ (by applying Theorem 1.1) to a selfdual, cuspidal automorphic representation Π of $\mathrm{GL}_a(\mathbb{A}_K)$ with π_τ as its local component at two places v, w of K with completions $K_v = k = K_w$.

Since π_τ is supercuspidal with central character of order ≤ 2, its parameter can be rendered trivial upon restriction to (the Weil group of) a finite extension of k which is a finite succession of cyclic extensions. Let E be a global extension of K, which is a finite succession of cyclic extensions, so that

(i) the place v splits completely in E;

(ii) the place w remains inert; and

(iii) the parameter of π_τ restricted to E_w is the trivial a-dimensional representation of W_k.

Let Π_E be the base change of Π to E (cf. [AC]). Let $\mathbb{B}$ be a central division algebra over E of dimension n^2 over E such that the invariant of $\mathbb{B}$ at a set of places of E above v of cardinality a is that of D, and such that $\mathbb{B}$ is ramified at the place w of E with Brauer invariant $\frac{-s}{b} \in \mathbb{Q}/\mathbb{Z}$, and unramified everywhere else. That there is such a division algebra $\mathbb{B}$ follows from classfield theory.

By [MW], Π_E gives rise to a selfdual representation in the residual spectrum of $\mathrm{GL}_{ab}(\mathbb{A}_E)$ denoted by $\Pi_E[b]$. Next, by Theorem 1.2 (of Badulescu), $\Pi_E[b]$ of $\mathrm{GL}_{ab}(\mathbb{A}_E)$ can be transported to an automorphic representation $\Pi' := \Pi'_E[b]$ of $\mathbb{B}^\times(\mathbb{A}_E)$, where $\mathbb{B}$ is the division algebra of index $n = ab$ over E just constructed.

By the product formula (Theorem C), $c(\Pi'_v)^a$ equals the sign associated to the representation Π'_w of $\mathrm{GL}_a(D')$ obtained by parabolic induction of the trivial representation of the minimal parabolic of $\mathrm{GL}_a(D')$ (which has parameter $(st_b)^a$), where D' is a division algebra over E_w with Brauer invariant $\frac{-s}{b}$. The representation Π'_w is clearly unitary, and defined over $\mathbb{R}$, hence orthogonal by Lemma 2.1. Thus $c(\Pi'_v) = 1$. This finishes the proof for $\mathbf{m} = \mathbf{1}$.

Now let us consider the case $\mathbf{m} > \mathbf{1}$. Let $M_m(D)$ have Brauer invariant $\frac{s}{d}$ with $(s,d) = 1$. Replacing E, if necessary, by a cyclic extension of degree $sm+1$ in which v splits completely, we may assume that there are places $v_0, v_1, \ldots, v_{sm}$ of E with $E_{v_j} = k$ such that for a selfdual, cusp form Π on $\mathrm{GL}_n(\mathbb{A}_E)$, $\Pi_{v_j} \simeq \pi$ for all $j \leq sm$. Now we may, by class field theory, choose a division algebra $\mathbb{D}$ ramified only at these v_j such that $\mathbb{D}_{v_0} \simeq M_m(D)$ of Brauer invariant s/d, and for every $j \in \{1, \ldots, sm\}$, $\mathbb{D}_{v_j} \simeq B^\times$, with B a division algebra over $E_{v_j} = k$ of Brauer invariant $-1/md$. Applying Theorem 1.2 again, we have a discrete automorphic representation $\Pi' := \Pi'_{\mathbb{D}}$ of $\mathbb{D}(\mathbb{A}_E)^\times$ whose local components have parameter σ at each of the v_j. Since $\mathbb{D}_{v_j}$ is a division algebra for each $j \geq 1$, we may apply what we proved above for $m = 1$ and conclude that $c(\Pi'_{v_j}) = 1$ for all $j \in \{1, \ldots, sm\}$. Thus, applying the product formula again, we obtain $c(\Pi'_v) = 1$, as desired.

To complete the proof in this case, we need to address the issue that the global Jacquet-Langlands correspondence might produce a Speh representation on $\mathrm{GL}_m(D)$, whereas the assertion of Theorem A is about a corresponding discrete series representations of $\mathrm{GL}_m(D)$. In other words, the local representation Π'_v of Π' may not be the π' we started with on $\mathrm{GL}_m(D)$ but is related to it by the Aubert-Zelevinsky involution. Nevertheless, we claim that

$$c(\pi') = c\big(\Pi'_v\big).$$

To deduce it we appeal to proposition 2.4 as well as the fact that the Aubert-Zelevinsky involution [Bad, Section 2.6] interchanges Speh-modules with the corresponding generalized Steinberg representations.

We are now finished with the proof of Theorem A, and also Corollary B, which is an immediate consequence. □

6. Rationality. The question whether an irreducible selfdual representation is orthogonal or symplectic is part of the more general question about *field of definition* of a representation. For example, as Lemma 2.1 shows, a selfdual, unitary representation of a p-adic group is orthogonal if and only if it is defined over $\mathbb{R}$. We discuss this more general question in the section.

Let G be a group, and π an irreducible representation of G over $\mathbb{C}$. Put

$$\mathcal{G}_\pi = \big\{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \mid \pi^\sigma \cong \pi\big\}.$$

If either G is finite, or G is a reductive p-adic group, and π is a discrete series representation with finite order central character, then $\mathcal{G}_\pi$ is known to be a subgroup of finite index of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ (as there are only finitely many isomorphism classes of discrete series representations, up to twisting by characters, which have a fixed vector under a given compact open subgroup of G as one knows that the formal degree of discrete series representations, up to twisting, tends to infinity).

The group $\mathcal{G}_\pi$ defines a finite extension $\mathcal{K}$ of $\mathbb{Q}$. Call $\mathcal{K}$ the field of definition of π. (If π is finite-dimensional, it is the field generated by the character values of π.)

Associated to π, there is a division algebra $\mathcal{D}_\pi$, finite dimensional over $\mathbb{Q}$, with center $\mathcal{K}$ (an abelian extension of $\mathbb{Q}$) which measures the obstruction to π being defined over $\mathcal{K}$, called the Schur algebra associated to π; thus $\mathcal{D}_\pi = \mathcal{K}$ if and only if π can be defined over $\mathcal{K}$. If π is finite-dimensional, these assertions are well-known; we leave the details for discrete series representations to the reader.

Now let $\sigma \to \pi_\sigma$ be the local Langlands correspondence between irreducible representations of $\mathrm{Gal}(\bar{k}/k)$ of dimension n, and irreducible discrete series representations of $\mathrm{GL}_m(D)$ where D is a division algebra of index d with $dm = n$. Here and in what follows, we normalize the Langlands correspondence by multiplying it by the character $x \to |x|^{(n-1)/2}$ where $x \in k^\times$. This normalized Langlands correspondence is what is Galois equivariant on the coefficients; see for example, Henniart [Hen2]. This implies that the center of $\mathcal{D}_\sigma$ is the same as the center of $\mathcal{D}_{\pi_\sigma}$, prompting us to ask the following question.

Question 6.1. How are $\mathcal{D}_\sigma$ and $\mathcal{D}_{\pi_\sigma}$ related?

The answer to Question 6.1 a priori might depend not just on the index of the division algebra, but on its class in the Brauer group. However, we propose the following conjecture, suggesting in particular that this is not the case.

CONJECTURE 6.1. *Let D be a division algebra of index d over a p-adic local field k, π an irreducible discrete series representation of $\mathrm{GL}_m(D)$, and σ the associated n-dimensional representation of the Weil-Deligne group of k for $n = md$. Let $\mathcal{D}_\sigma$ and $\mathcal{D}_\pi$ be the associated Schur algebras with center a number field $\mathcal{K}$ (which is a cyclotomic field). For the algebras $\mathcal{D}_\sigma$ and $\mathcal{D}_\pi$, the following hold:*

(1) *If π is not selfdual, or if π is selfdual with $c(\sigma) = c(\pi)$, then $\mathcal{D}_\sigma = \mathcal{D}_\pi$.*

(2) *If π is selfdual and $c(\sigma) = -c(\pi)$, then $\mathcal{K}$ is totally real and the answer depends on the parity of the degree of $\mathcal{K}$ over $\mathbb{Q}$. If $[\mathcal{K} : \mathbb{Q}]$ is even, then the Brauer invariants of $\mathcal{D}_\sigma$ and $\mathcal{D}_\pi$ are the same at all the finite places of $\mathcal{K}$, and at all the infinite places the invariants of $\mathcal{D}_\sigma$ and $\mathcal{D}_\pi$ differ by $1/2$. If $[\mathcal{K} : \mathbb{Q}]$ is odd, in which case there is an odd number of places in $\mathcal{K}$ above p, the invariants of $\mathcal{D}_\sigma$ and $\mathcal{D}_\pi$ are the same except at the archimedean places and the places in $\mathcal{K}$ above p, where the invariants of $\mathcal{D}_\sigma$ and $\mathcal{D}_\pi$ differ by $1/2$.*

Remark. The proof of Harris-Taylor [HT] realizing the Langlands correspondence between $\mathrm{GL}_n(k)$, $D^\times$, and the Weil group on an ℓ-adic cohomology implies that the Schur algebras $\mathcal{D}_\sigma$ and $\mathcal{D}_\pi$ are the same expect at the places above p and infinity. Since the sum of local invariants of a central simple algebra over a number field is zero, the information at infinity contained in this paper determines the relationship between $\mathcal{D}_\sigma$ and $\mathcal{D}_\pi$ when $\mathcal{K}$ has odd degree after one notes the well-known theorem due to Benard-Schacher, cf. the book of Curtis-Reiner, volume II, page 746, that the invariants of these algebras at places v of $\mathcal{K}$ over a place q of $\mathbb{Q}$ are related to each other in a simple way. The case of $[\mathcal{K} : \mathbb{Q}]$ even remains open.

One case of the conjecture is especially simple to state. This is when d, the index of D, is odd, and π is a selfdual representation of $\mathrm{GL}_m(D)$ with m odd. In this case, irreducible selfdual representations of the Galois group of dimension n exist only in even residual characteristic, cf. [Pra]. We are thus in the tame case $(n, p) = 1$, and here it can be seen that such a Galois representation is induced by a character θ of $L^\times$, where L is a degree n extension of k, with $\theta^2 = 1$. Thus θ takes values ± 1, and it follows in this case that the Galois representation is defined over $\mathbb{Q}$. Our Theorem B implies that the selfdual representations of $\mathrm{GL}_m(D)$ are defined over $\mathbb{R}$, and the discussion in this section refines it to ask the following.

Question 6.2. Let D be a division algebra of odd index over a non-archimedean local field k, and $m > 0$ an odd integer. Then, is every selfdual, irreducible discrete series representation of $\mathrm{GL}_m(D)$ defined over $\mathbb{Q}$?

Recently, Bushnell and Henniart have answered Question 6.2 in the affirmative in [BH2] if either $m = 1$, or $d = 1$.

7. A concluding remark. One of the reasons the proof of Theorem A is so involved is that we do not know, as of yet, how to simultaneously globalize to a selfdual cusp form on $\mathrm{GL}(n)$ a finite number of selfdual, square-integrable local representations (with parameters of the same parity), when one of the representations is supercuspidal while another is (generalized) Steinberg. The reason is that the method of Poincaré series applied to the classical groups will produce a generic cusp form with appropriate local components *only when* every local discrete series representation is integrable, which is not the case for the Steinberg representation and its generalizations.

It is expected that such a general globalization result will follow by an application, yet to be carried out, of Arthur's stabilization of the twisted trace formula for $\mathrm{GL}(n)$ in "Endoscopic Classification of Representations: Orthogonal and Symplectic groups" (http://www.claymath.org/cw/arthur/pdf/Book.pdf). Some partial results can be found in [CC].

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REFERENCES

[AC] J. Arthur and L. Clozel, *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, *Ann. of Math. Stud.*, vol. 120, Princeton University Press, Princeton, NJ, 1989.

[Bad] A. I. Badulescu, Global Jacquet-Langlands correspondence, multiplicity one and classification of automorphic representations. With an appendix by Neven Grbac, *Invent. Math.* **172** (2008), no. 2, 383–438.

[BH1] C. J. Bushnell and G. Henniart, Explicit functorial correspondences for level zero representations of p-adic linear groups, *J. Number Theory* **131** (2011), no. 2, 309–331.

[BH2] ———, Self-dual representations of some dyadic groups, *Math. Ann.* **351** (2011), no. 1, 67–80.

[CC] G. Chenevier and L. Clozel, Corps de nombres peu ramifiés et formes automorphes autoduales, *J. Amer. Math. Soc.* **22** (2009), no. 2, 467–519.

[HT] M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties. With an Appendix by Vladimir G. Berkovich*, *Ann. of Math. Stud.*, vol. 151, Princeton University Press, Princeton, NJ, 2001.

[Hen1] G. Henniart, Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique, *Invent. Math.* **139** (2000), no. 2, 439–455.

[Hen2] ———, Une caractérisation de la correspondance de Langlands locale pour $\mathrm{GL}(n)$, *Bull. Soc. Math. France* **130** (2002), no. 4, 587–602.

[Hen3] ———, Correspondence de Langlands et fonctions L des carrés extérieur et symétrique, *Internat. Math. Res. Notices* (2010), no. 4, 633–673.

[MW] C. Mœglin and J.-L. Waldspurger, Le spectre résiduel de $\mathrm{GL}(n)$, *Ann. Sci. École Norm. Sup. (4)* **22** (1989), no. 4, 605–674.

[Pra] D. Prasad, Some remarks on representations of a division algebra and of the Galois group of a local field, *J. Number Theory* **74** (1999), no. 1, 73–97.

[PR] D. Prasad and D. Ramakrishnan, Lifting orthogonal representations to spin groups and local root numbers, *Proc. Indian Acad. Sci. Math. Sci.* **105** (1995), no. 3, 259–267.

[SZ] A. J. Silberger and E.-W. Zink, An explicit matching theorem for level zero discrete series of unit groups of $\mathfrak{p}$-adic simple algebras, *J. Reine Angew. Math.* **585** (2005), 173–235.

[Vog] D. A. Vogan, Jr., The algebraic structure of the representation of semisimple Lie groups. I, *Ann. of Math. (2)* **109** (1979), no. 1, 1–60.

[Zel] A. V. Zelevinsky, Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$, *Ann. Sci. École Norm. Sup. (4)* **13** (1980), no. 2, 165–210.

[Žel1] D. P. Želobenko, *Compact Lie groups and their representations*, *Transl. Math. Monogr.*, vol. 40, American Mathematical Society, Providence, RI, 1973, Translated from the Russian by Israel Program for Scientific Translations.

[Žel2] ———, *Harmonic Analysis on Semisimple Complex Lie Groups*, Izdat. "Nauka", Moscow, 1974.

Appendix: On the local descent from $\mathrm{GL}(n)$ to classical groups

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Introduction. The descent method of Ginzburg, Rallis and Soudry enables one to construct, for an irreducible, self-dual, automorphic, cuspidal representation τ of $\mathrm{GL}_m(\mathbb{A})$, with $\mathbb{A}$ being the Adele ring of a number field k, an irreducible, automorphic, cuspidal and globally generic representation σ on the Adelic points of the appropriate orthogonal (split or quasi-split), or symplectic group G, or metaplectic group (which we also denote by $G(\mathbb{A})$) such that σ lifts to τ at almost all places. See [GRS1, GRS4, GRS3, GRS6, GRS5, S2]. This method works also for a representation (cuspidal, generic) σ of a quasi-split unitary group $G(\mathbb{A}) = \mathrm{U}_m(\mathbb{A})$, associated to a quadratic extension E of k, and its lift τ on $\mathrm{GL}_m(\mathbb{A}_E)$. In case $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$, and hence $m = 2n$ is even, we showed that σ lifts to τ at all places [JS1, JS2]. This was done by local descent, which is the local counterpart of the global descent method, with almost complete analogy. It allows us to construct, when G is orthogonal, symplectic, or metaplectic, for an irreducible, self-dual, supercuspidal representation τ of $\mathrm{GL}_m(F)$, where F is a p-adic field, an irreducible, supercuspidal, generic representation σ of $G(F)$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$, or equivalently, $L(\sigma \times \tau, s)$ has a pole at $s = 0$. This works also for quasi-split unitary groups. See [GRS3], where the representations (irreducible, supercuspidal) τ of $\mathrm{GL}_{2n}(F)$, whose local exterior square L-function has a pole at $s = 0$, are treated. Here, the representation σ, as above, is on the metaplectic group $\widetilde{\mathrm{Sp}}_{2n}(F)$. The case of even quasi-split unitary groups is treated in [T]. In this note, we will present a similar result for an irreducible, supercuspidal representation τ of $\mathrm{GL}_m(F)$, whose symmetric square L-function has a pole at $s = 0$. In this case, $G = \mathrm{Sp}_{2n}$, when $m = 2n+1$, and $G = \mathrm{SO}_{2n}$, when $m = 2n$ (split, or quasi-split over F, according to the central character of τ being trivial or (nontrivial) quadratic).

Local gamma factors. Let F be a p-adic field, and $G = G(F)$ be a local orthogonal, symplectic, or metaplectic group, as in the introduction. Let σ, τ be irreducible, generic representations of G, $\mathrm{GL}_m(F)$, respectively. The local gamma factor $\gamma(\sigma \times \tau, s, \psi)$ (ψ is a nontrivial character of F) is obtained via a local functional equation, which arises from the theory of global integrals of Rankin-Selberg type, or Shimura type, and represent the standard L-functions for $G \times \mathrm{GL}_m$. See [G, GRS2, GRS3, S1], for G odd orthogonal, symplectic, or metaplectic. The case where G is even orthogonal is treated in [G, K]. We restrict ourselves to $\mathrm{rank}(G) < m$. The local functional equation has the form

$$\frac{\gamma(\sigma \times \tau, s, \psi)}{c(\tau, s, \psi)} \mathcal{L}\big(W_\sigma, D^\psi\big(f_{\tau,s}\big)\big) = \mathcal{L}\big(W_\sigma, D^\psi\big(M\big(f_{\tau,s}\big)\big)\big),$$

where $\mathcal{L}(W_\sigma, -)$ is defined below, following the table. Here, W_σ is in the Whittaker model of σ (with respect to a given character), $f_{\tau,s}$ is a holomorphic section in $\rho_{\tau,s} = \mathrm{Ind}_P^H \tau|\det\cdot|^{s-\frac{1}{2}}$, where H is an appropriate split classical group, or a metaplectic group, and $P \subset H$ is a Siegel type parabolic subgroup, with Levi part isomorphic to GL_m, according to the following table, where we also specify $c(\tau, s, \psi)$, and where ρ signifies the relevant finite-dimensional representation of $\mathrm{GL}_m(\mathbb{C})$.

	G	H	$c(\tau,s,\psi)$	ρ
1.	$\mathrm{SO}_{2n+1}(F)$	$\mathrm{SO}_{2m}(F)$	$\gamma(\tau,\Lambda^2,2s-1,\psi)$	Λ^2
2.	$\widetilde{\mathrm{Sp}}_{2n}(F)$	$\mathrm{Sp}_{2m}(F)$	$\gamma(\tau,\Lambda^2,2s-1,\psi)\,\gamma\left(\tau,s-\frac{1}{2},\psi\right)$	$\Lambda^2\oplus st$
3.	$\mathrm{SO}_{2n}(F)$	$\mathrm{SO}_{2m+1}(F)$	$\gamma(\tau,\mathrm{sym}^2,2s-1,\psi)$	sym^2
4.	$\mathrm{Sp}_{2n}(F)$	$\widetilde{\mathrm{Sp}}_{2m}(F)$	$\gamma(\tau,\mathrm{sym}^2,2s-1,\psi)$	sym^2

In case (4), we have to consider $\rho_{\tau,s}=\mathrm{Ind}_P^H\gamma_\psi\tau|\det\cdot|^{s-\frac{1}{2}}$ instead (γ_ψ is the Weil factor); M is the intertwining operator corresponding to the long Weyl element;

$$\mathcal{L}\big(W_\sigma,D^\psi\big(f_{\tau,s}\big)\big)=\int_{N\backslash G}W_\sigma(g)D^\psi\big(f_{\tau,s}\big)(g)dg,$$

where N is the (standard) maximal unipotent subgroup of G; $D^\psi(f_{\tau,s})$ is given as an integral along the unipotent radical of the standard parabolic subgroup, which preserves a maximal flag in a totally isotropic subspace of dimension ℓ, and factors through the Jacquet module of $\rho_{\tau,s}$, which furnishes a Gelfand-Graev (resp. Fourier-Jacobi) model of $\rho_{\tau,s}$, stabilized by G, when H is orthogonal (resp. symplectic or metaplectic). In fact D^ψ defines an isomorphism with this Jacquet module when τ is supercuspidal. Denote this Jacquet module by $\sigma_{\psi,\ell}(\rho_{\tau,s})$. The number ℓ is determined easily by n and m; it is $m-n-1$, in cases (1), (3), (4), and it is $m-n$, in case (3). The gamma factor thus defined is the same as the Shahidi gamma factor, at least up to a multiple by an exponential function, and hence it has the same set of poles and zeroes.

Descent. The local functional equation above, defining the local gamma factor, implies the following theorem.

THEOREM A1. *Let σ, τ be irreducible, supercuspidal representations of G, $\mathrm{GL}_m(F)$ respectively. Then $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$, if and only if $L(\tau,\rho,s)$ has a pole at $s=0$ and σ pairs with the Jacquet module above, where we replace $\rho_{\tau,1}$ with its image π_τ by the intertwining operator M at $s=1$.*

Since $L(\tau,\rho,s)$ has a pole at $s=0$, π_τ is the Langlands quotient of $\rho_{\tau,1}$. We call $\sigma_{\psi,\ell}(\tau)=\sigma_{\psi,\ell}(\pi_\tau)$ the descent of τ to G. Consider the cases of functoriality.

	$\mathrm{GL}_m(F)$	*pole at* $s=0$	H	G	*descent*
1.	$\mathrm{GL}_{2n}(F)$	$L(\tau,\Lambda^2,s)$	$\mathrm{SO}_{4n}(F)$	$\mathrm{SO}_{2n+1}(F):\ell=n-1$	$\sigma_{\psi,n-1}(\tau)$
2.	$\mathrm{GL}_{2n}(F)$	$L(\tau,\Lambda^2,s)$	$\mathrm{Sp}_{4n}(F)$	$\widetilde{\mathrm{Sp}}_{2n}(F):\ell=n-1$	$\sigma_{\psi,n-1}(\tau)$
3.	$\mathrm{GL}_{2n}(F)$	$L(\tau,\mathrm{sym}^2,s)$	$\mathrm{SO}_{4n+1}(F)$	$\mathrm{SO}_{2n,\alpha}(F):\ell=n$	$\sigma_{\psi,n,\alpha}(\tau),\omega_\tau=\chi_\alpha$
4.	$\mathrm{GL}_{2n+1}(F)$	$L(\tau,\mathrm{sym}^2,s)$	$\widetilde{\mathrm{Sp}}_{4n+2}(F)$	$\mathrm{Sp}_{2n}(F):\ell=n$	$\sigma_{\psi,n}(\tau),\omega_\tau=1$

Here, ω_τ is the central quadratic character of τ. If it corresponds to $\alpha\in F^*$, we denote it also by χ_α, and then we denote by $\mathrm{SO}_{2n,\alpha}(F)$ the corresponding quasi-split (or split when α is a square) orthogonal group in $2n$ variables. We can prove the following theorem.

THEOREM A2. *In all these cases, the descent of τ is a nontrivial, supercuspidal, multiplicity free representation of G, all of whose irreducible summands σ are ψ-generic and are such that $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$. Moreover, any irreducible, supercuspidal and ψ-generic representation σ of G, such that $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$ is isomorphic to a summand of the descent of τ.*

The proof in cases (1), (2) appears in [GRS3, JS1]. The proof in cases (3), (4) will appear in detail in a forthcoming paper by the authors. It uses the tower property of local descents (see [GRS3, Section 2]), local gamma factors, the theory of global descent, summarized in [S2]; it will appear in great detail in [GRS7]. The proof uses also the existence of the weak functorial lift from cuspidal generic representations of $G(\mathbb{A})$ to $\mathrm{GL}_m(\mathbb{A})$ (appropriate m), proved in [CKPSS].

Assume that F is completion at a place v of the number field k. Consider a self-dual, supercuspidal τ, as above, and an irreducible summand σ of its descent to G. By globalizing σ to an irreducible, automorphic, cuspidal, generic representation of $G(\mathbb{A})$, and lifting it to $\mathrm{GL}_m(\mathbb{A})$, we get

THEOREM A3. *Let τ be an irreducible, self-dual, supercuspidal representation of* $\mathrm{GL}_m(F)$. *Assume that $L(\tau,\rho,s)$ has a pole at $s=0$, where $\rho=\Lambda^2$,* sym^2. *Then we can globalize τ to an irreducible, self-dual, automorphic, cuspidal representation T of* $\mathrm{GL}_m(\mathbb{A})$*, such that $L(T,\rho,s)$ has a pole at $s=1$.*

The same argument shows

THEOREM A4. *Let $v_1,\ldots,v_r$ be r finite places of the number field k. Let $\tau_1,\ldots,\tau_r$ be r irreducible, self-dual, supercuspidal representations of* $\mathrm{GL}_m(k_{v_1}),\ldots,\mathrm{GL}_m(k_{v_r})$*, respectively. Assume that $L(\tau_i,\rho,s)$ has a pole at $s=0$, for all $i\le r$, where $\rho=\Lambda^2$,* sym^2. *Then we can globalize $\tau_1\otimes\cdots\otimes\tau_r$ to an irreducible, self-dual, automorphic, cuspidal representation T of* $\mathrm{GL}_m(\mathbb{A})$*, such that $L(T,\rho,s)$ has a pole at $s=1$.*

We simply pick irreducible summands $\sigma_1,\ldots,\sigma_r$ in the local descents of $\tau_1,\ldots,\tau_r$, respectively, globalize $\sigma_1\otimes\cdots\otimes\sigma_r$ to an irreducible, automorphic, cuspidal, generic representation, and lift it to $\mathrm{GL}_m(\mathbb{A})$, as before. In case (2) of the last table, the local descent is irreducible. This is proved in [GRS3]. Using this and the local theta correspondence, we proved that the descent is irreducible in case (1), as well. See [JS1]. This means that σ in the last theorem is unique. For a long time we tried to address the irreducibility question of the descent in cases (3), (4), without success. Here is our new idea. Let us add two more cases to the last table:

	$\mathrm{GL}_m(F)$	*pole at* $s=0$	H	G	*descent*
5.	$\mathrm{GL}_{2n}(F)$	$L(\tau,\mathrm{sym}^2,s)$	$\widetilde{\mathrm{Sp}}_{4n}(F)$	$\mathrm{Sp}_{2n}(F):\ell=n-1$	$\sigma'_{\psi,n-1}(\tau)$
6.	$\mathrm{GL}_{2n+1}(F)$	$L(\tau,\mathrm{sym}^2,s)$	$\mathrm{SO}_{4n+3}(F)$	$\mathrm{SO}_{2n+2}(F):\ell=n$	$\sigma'_{\psi,n},\ \omega_\tau=1$

Here, we denote the descent by $\sigma'_{\psi,\ell}$, in order to distinguish it from the one in cases (4), (5). We can prove:

THEOREM A5. *Let τ be an irreducible, supercuspidal representation of $\mathrm{GL}_m(F)$, such that $L(\tau, \mathrm{sym}^2, s)$ has a pole at $s = 0$. In case m is odd, assume that $\omega_\tau = 1$. Then the descent in cases (5), (6) above is a nonzero irreducible, supercuspidal, ψ-generic representation σ of G, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$; these properties determine σ uniquely.*

Consider case (5) and denote $\sigma = \sigma'_{\psi,n-1}(\tau)$. The local lift of σ to $\mathrm{GL}_{2n+1}(F)$ must be $\tau \times \omega_\tau$. We conclude that $L(\sigma \times \omega_\tau, s)$ has a pole at $s = 0$. We then prove that σ is the local ψ-theta lift from an irreducible, supercuspidal, ψ-generic representation π of $\mathrm{O}_{2n,\alpha}(F)$, where $\omega_\tau = \chi_\alpha$. We know that the restriction of π to $\mathrm{SO}_{2n,\alpha}(F)$ is either irreducible or a direct sum of two irreducible representations of the form $\pi_1 \oplus \pi_1^\varepsilon$, where $\varepsilon \in \mathrm{O}_{2n,\alpha}(F)$, with $\det(\varepsilon) = -1$; π_1^ε denotes the outer conjugation of π_1 by ε. Thus,

THEOREM A6. *Let τ be an irreducible, supercuspidal representation of $\mathrm{GL}_{2n}(F)$, such that $L(\tau, \mathrm{sym}^2, s)$ has a pole at $s = 0$. Let $\omega_\tau = \chi_\alpha$. Then there is an irreducible, supercuspidal, ψ-generic representation σ of $\mathrm{SO}_{2n,\alpha}(F)$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$, and it is unique up to outer conjugation by ε.*

Similarly, in case (6), we get:

THEOREM A7. *Let τ be an irreducible, supercuspidal, self-dual representation of $\mathrm{GL}_{2n+1}(F)$, with $\omega_\tau = 1$. Then there is a unique irreducible, supercuspidal, ψ-generic representation σ of $\mathrm{Sp}_{2n}(F)$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$.*

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REFERENCES

[CKPSS] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, Functoriality for the classical groups, *Inst. Hautes Études Sci. Publ. Math.* (2004), no. 99, 163–233.

[G] D. Ginzburg, L-functions for $\mathrm{SO}_n \times \mathrm{GL}_k$, *J. Reine Angew. Math.* **405** (1990), 156–180.

[GRS1] D. Ginzburg, S. Rallis, and D. Soudry, Self-dual automorphic GL_n modules and construction of a backward lifting from GL_n to classical groups, *Internat. Math. Res. Notices* (1997), no. 14, 687–701.

[GRS2] ———, L-functions for symplectic groups, *Bull. Soc. Math. France* **126** (1998), no. 2, 181–244.

[GRS3] ———, On a correspondence between cuspidal representations of GL_{2n} and $\widetilde{\mathrm{Sp}}_{2n}$, *J. Amer. Math. Soc.* **12** (1999), no. 3, 849–907.

[GRS4] ———, On explicit lifts of cusp forms from GL_m to classical groups, *Ann. of Math. (2)* **150** (1999), no. 3, 807–866.

[GRS5] ———, Generic automorphic forms on $\mathrm{SO}(2n+1)$: functorial lift to $\mathrm{GL}(2n)$, endoscopy, and base change, *Internat. Math. Res. Notices* (2001), no. 14, 729–764.

[GRS6] ———, Endoscopic representations of $\widetilde{\mathrm{Sp}}_{2n}$, *J. Inst. Math. Jussieu* **1** (2002), no. 1, 77–123.

[GRS7] ———, *The Descent Map from Automorphic Representations of* $\mathrm{GL(n)}$ *to Classical Groups*, World Scientific Publishing, Hackensack, NJ, 2011.

[JS1] D. Jiang and D. Soudry, The local converse theorem for $\mathrm{SO}(2n+1)$ and applications, *Ann. of Math. (2)* **157** (2003), no. 3, 743–806.

[JS2] ———, Generic representations and local Langlands reciprocity law for p-adic SO_{2n+1}, *Contributions to Automorphic Forms, Geometry, and Number Theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 457–519.

[K] E. Kaplan, On the local Rankin-Selberg convolutions of $\mathrm{SO}(2n) \times \mathrm{GL}(m)$, Thesis, Tel-Aviv University, 2012.

[S1] D. Soudry, Rankin-Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$: local theory, *Mem. Amer. Math. Soc.* **105** (1993), no. 500, vi+100.

[S2] ———, On Langlands functoriality from classical groups to GL_n. Automorphic forms. I, *Astérisque* (2005), no. 298, 335–390.

[T] J. Tanai, On a correspondence between cuspidal representations of $\mathrm{GL}(2n)$ and $\mathrm{U}(2n)$, Thesis, Tel-Aviv University, 2012.