1 Introduction

The number line is foundational to present day mathematics, and in a broader way it is a major marker of the progress of human civilization. From the early beginnings of counting on fingers we have come a long way, developing a system with which one can grasp the intricate laws of the physical universe.

The path however has been quite tortuous. From natural numbers, especially their infinitude, fractions, the zero, negative numbers have each been a big conceptual stride, taken over long historical periods. The difficulty in a concept like fraction being grasped in full by the human mind may be realised if one recalls how a significant proportion of our students have difficulty in adding them. The ancient Indian civilization contributed greatly to the rise of zero; it was however not invented one fine day by some sage, as popular imagination would have it, but had a long and chequered history involving several ancient civilizations\[8\]. Negative numbers must have been lurking in the background for long. But a systematic approach to them appears for the first time in the work of Brahmagupta (589-668).

Notwithstanding some lacunae, various ancient civilizations may be said to have had a general understanding of fractions, or what we now call rational numbers, over two thousand years ago. Going beyond them however posed a bigger challenge. Indeed, various numbers that we now know to be irrational were encountered in early times primarily in various geometric contexts. The Sulvasutras, which are compositions from the Vedangas (appendages of the Vedas) describing construction of various fire alters and various geometric principles involved in the constructions, contain the following value for the

\[
\sqrt{2} \approx 1.414213562373095\]

\[\text{Text version of the “T.B. Hardikar Memorial Lecture” given at S.P. College, Pune, on 25 August 2007}\]
proportion of the diagonal of a square to its side, viz. the square root of 2, that we write as \( \sqrt{2} \):

\[
1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34};
\]

(see [13] for an exposition on Sulvasutras.) The period of the Sulvasutras is difficult to ascertain, but is generally accepted to be between 800-400 BCE (Before Christian Era), with the earliest, the Baudhayana Sulvasutra near the early end of it. A cuneiform tablet (Yale Babylonian collection, No. 7289) from the old Babylonian period (around 1800-1600 BCE) shows a square with 1, 24, 51, 10 written across the diagonal; in the sexagesimal system (with base 60, in the place of 10 that we use) which the Babylonians used the number stands for

\[
1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3};
\]

(it also stands for the multiples of the above by powers of 60, but it does not concern us here). It is quite striking that the above values for \( \sqrt{2} \) adopted by the Sutrakaras and Babylonians match with the value of \( \sqrt{2} \) up to five decimals, 1.4142156... and 1.4142129... respectively, in the place of 1.4142135.... (These are however unique instances in their respective contexts, and what led to the strikingly close values in these cases is not clear.)

Sulvasutras are also concerned with transforming a square to a circle (disc) of equal area, and vice versa. For the former the construction is geometric (and the equality of areas is approximate). For the latter there is a curious formula prescribing for the ratio of the side of a square to the diameter of a circle with equal area, the number

\[
1 - \frac{1}{8} + \frac{1}{8 \times 29} - \frac{1}{8 \times 29} \left( \frac{1}{6} - \frac{1}{6 \times 8} \right).
\]

As in the case of \( \sqrt{2} \), the actual number involved here is irrational, and the value represents an approximation, adopted for practical purposes. The ancient Egyptians also concerned themselves with the area of a circle, which they took to be \((8/9)^2\) times the square of the diameter [7].

The first systematic approach going beyond the rationals was introduced by the Greek mathematician Eudoxus of Cnidos (400-347 BCE), known as the Eudoxus’s theory of proportions. In geometry, which was their main strength, the Greek mathematicians encountered magnitudes of lengths which they recognised to be “incommensurable”. They had philosophical reservations
about treating these magnitudes as numbers. Nevertheless they had a theory for the magnitudes, akin to the present day concept of real numbers. According to Eudoxus’ theory ratio of magnitudes \(a\) and \(b\) is equal to that of magnitudes \(c\) and \(d\) if for positive integers \(m\) and \(n\), \(ma > nb\) we also have \(mc > nd\) (see [3], page 89). If we choose \(c\) and \(d\) to be unit magnitudes this reduces to that \(a\) and \(b\) are equal if \(ma > nb\) for all positive integers \(m, n\) with \(m > n\). (Perhaps it calls for some explanation why one defines equality, whereas the relations < and > are being assumed. In this respect, without going into details let me only mention that this has to do with verifiability of the statements in terms of known, i.e. “commensurable”, magnitudes.)

Crystallisation of the idea of real numbers was however to wait for over two thousand years. With the rise of calculus, and the idea of infinitesimals, in the seventeenth and eighteenth centuries the intuitive sense of continuity of the number line caught hold. Since the number line was now seen as being “continuous”, a rigorous way to “fill in” between rationals was needed. An elegant definition of real numbers fulfilling this need was introduced by Dedekind in 1858. The idea involved is that for a real number as we conceive it intuitively, the collection of rational numbers less than that is a characteristic feature of the number. Taking this into account Dedekind defined real numbers as partitions \((L, U)\) of the set of rational numbers, with the property that for any \(r\) in \(L\) and \(s\) in \(U\), \(r < s\); \((L, U)\) being a partition means that \(L\) and \(U\) are disjoint sets of rationals which together cover all rationals). Such a partition is called a Dedekind cut. Thus for example \(\sqrt{2}\) is to be thought of (by definition) as the Dedekind cut \((L, U)\) with \(L\) and \(U\) subsets of rationals consisting of \(\{r \mid r \leq 0 \text{ or } r^2 < 2\}\) and \(\{r \mid r > 0 \text{ and } r^2 \geq 2\}\) respectively. A rational number \(q\) may be thought of as the Dedekind cut into \(\{r \mid r < q\}\) and \(\{r \mid r \geq q\}\).

One can then extend the operations of addition and multiplication to the set of real numbers so defined, in a natural way (that the reader is encouraged to find for herself). Furthermore it can be verified that every positive real number has a positive square root, cube root etc.

Dedekind’s definition greatly facilitates verification of arithmetical statements about real numbers. This is dramatised in the title of [6]: Dedekind’s theorem: \(\sqrt{2} \times \sqrt{3} = \sqrt{6}\). While this may seem facetious (as the statement seems obvious) it may be noticed that writing a rigorous argument poses a variety of problems, including the matter of definition itself, which are readily taken care of by Dedekind’s approach. The reader is referred to [6] for a discussion on how various ways thinking of real numbers through various systems of labelling, such as decimal representation or continued fractions.
(these we shall discuss below) do not lend themselves to the possibility of proving the simple theorem as above. For example the decimal expansions of $\sqrt{2}$ and $\sqrt{3}$ are infinite there is no way to write what the product of two numbers expressed in this way, (since as we go down the decimal places the products of the corresponding digits would in general exceed 10, and keep calling for alteration in the digits in earlier places in a never ending way).

There are of course alternative possibilities for constructing the number line. The real numbers can also be thought of as limits of Cauchy sequences of rational numbers. However since different Cauchy sequences can have the same limit, an individual real number needs to be thought of as “equivalence class” under an equivalence relation which identifies two such Cauchy sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers if the difference $a_n - b_n$ tends to 0. This can indeed be converted to a definition of real numbers and the arithmetic operations on real numbers can be introduced through them. The idea of considering equivalence classes as elements is however a late nineteenth century stratagem, that came into vogue long after Dedekind cuts. Besides even today the Dedekind cuts provide an alternative which is simpler in various ways. On the other hand the idea of using Cauchy sequence as above has other applications, such as for instance in constructing what are called $p$-adic numbers, for any prime number $p$.

## 2 Approximating real numbers by rationals

We normally view the number line as being “uniform”, looking the same everywhere, so it would seem there is no “structure” to speak of, and the title of the article may seem confusing. With some reflexion it would be clear that the intuitive sense of uniformity is the consequence of the fact that we always think of it in terms of equal subdivisions, say in terms of decimal expansion, binary expansion etc. This uniformity is of course in terms of analysis. It is however superposed, and need not be viewed entirely as constituting the structure of the number line. Recall that the number line was constructed from rational numbers, by filling in more numbers in between. In the equal subdivisions as above all rationals do not feature. To think of the structure we may think of how numbers get filled in between rational numbers, or in other words how the real numbers can be approximated rational numbers. This would involve all rational numbers rather than binary or decimal rationals. I will discuss various results on approximations. This aspect is what I have referred to in the title as the fine structure of the number line.

Before going into what I mean, let me recall what are called Farey frac-
A Farey fraction of order $k$ is a rational number $\frac{p}{q}$ between 0 and 1, with $q \leq k$. We shall consider these arranged in the increasing order. The first five rows are as shown below; (we shall write all the fractions are written in the reduced form).

\[
\begin{array}{c|ccc}
0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 2 & 1 \\
1 & 3 & 2 & 3 & 1 \\
0 & 1 & 4 & 3 & 2 & 3 & 4 & 1 \\
1 & 5 & 4 & 3 & 2 & 3 & 4 & 5 & 1 \\
\end{array}
\]

It turns out that the entries in each successive row may be obtained by inserting the fraction $\frac{m+p}{n+q}$ between consecutive entries $\frac{m}{n}$ and $\frac{p}{q}$ of the previous row, whenever $n + q$ does not exceed the number of the row being written down. This gives an alternative definition of the Farey fractions in the successive rows.

The Farey fractions have interesting properties; see [9] for instance. If $\frac{m}{n}$ and $\frac{p}{q}$ are successive entries in the $k$th row, then $np - mq = 1$ and $n + q \geq k + 1$. Now consider any real number $\alpha$ between 0 and 1 and let $\frac{m}{n}$ and $\frac{p}{q}$ be successive entries in the $k$th row such that $\frac{m}{n} \leq \alpha \leq \frac{p}{q}$. Then either $\frac{m}{n} \leq \alpha \leq \frac{m+p}{n+q}$ or $\frac{m+p}{n+q} \leq \alpha \leq \frac{p}{q}$, and together with the above this implies that either $|\alpha - \frac{m}{n}| \leq \frac{1}{(k+1)n}$ or $|\alpha - \frac{p}{q}| \leq \frac{1}{(k+1)q}$. Thus we have the following.

**Theorem 2.1.** Let $\alpha$ be a real number between 0 and 1 and $\frac{p}{q}$ be the Farey fraction of order $k$ at least possible distance from $\alpha$. Then

$$|\alpha - \frac{p}{q}| \leq \frac{1}{(k+1)q}.$$  

As a consequence we get the following:

**Corollary 2.2.** Let $\alpha$ be an irrational real number. Then there exist infinitely many rational numbers $\frac{p}{q}$ such that

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$  

5
This does not hold for rational $\alpha$; if $\alpha = \frac{m}{n}$ the inequality as above can hold only for $\frac{p}{q}$ with $q \leq n$, which are only finitely many. Note that when there are infinitely many rationals satisfying an inequality they provide closer and closer approximation of $\alpha$ satisfying the restriction on the denominators.

Better approximations are possible through the study of “continued fractions”. We briefly recall here the concept and some properties of continued fractions; the reader is referred to [9] and [11] for further details. Let $\alpha$ be a real number. If $\alpha$ is not an integer it can be written as $m_0 + \frac{1}{\alpha_1}$ with $\alpha_1 > 1$. Now unless $\alpha_1$ is an integer it may be written as $m_1 + \frac{1}{\alpha_2}$ with $\alpha_2 > 1$, and the process may be repeated, until either we reach an integer, or indefinitely. It can be seen that the former happens if $\alpha$ is a rational and the latter when $\alpha$ is irrational. Thus any irrational number $\alpha$ can be written as

$$m_0 + \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_k + \frac{1}{\cdots}}}}$$

where $m_0$ is an integer and $m_k, k \geq 1$ are positive integers, and every rational number has such an expansion which stops at some $k$. To avoid writing the cumbersome, though illustrative, expression as above we shall denote it by $[m_0, m_1, \ldots, m_k, \ldots]$. Similarly we shall denote by $[m_0, m_1, \ldots, m_k]$ the analogous but terminating expression (omitting the part after $m_k$ in the above expression. If $[m_0, m_1, \ldots, m_k, \ldots]$ is the representation as above for $\alpha$ we shall $[m_0, m_1, \ldots, m_k]$ by $\alpha_k$, which is a rational number. A crucial thing is that the sequence $\alpha_k$ converges to $\alpha$; this is what makes it meaningful to express $\alpha$ as above (otherwise the infinite expression makes no sense by itself). The expansion as above is called the continued fraction expansion of $\alpha$. Conversely, given an integer $m_0$ and positive integers $m_k, k \geq 1$, the expression as above is the continued fraction expansion of a unique irrational number $\alpha$. Note that unlike decimal or binary expansions, the continued fraction expansions of numbers are intrinsic, and do not depend on any ad hoc choice of basis.

The numbers $m_k$ as above, associated with $\alpha$, which we shall henceforth take to be irrational, are called partial quotients and the rationals $\alpha_k$ as above are called the convergents of $\alpha$. The $\alpha_k, k \geq 1$ are best approximations for $\alpha$ in the sense that if $\alpha_k = \frac{p_k}{q_k}$ then $\alpha_k$ is at least as close to $\alpha$ as any rational with denominator not exceeding $q_k$ (namely any Farey fraction of
order $q_k$), when $0 < \alpha < 1$. We thus have, by Theorem 2.1 $|\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k^2}$. Conversely, if $\frac{p}{q}$ is a rational for which the slightly stronger approximation $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$ holds, then it is necessarily one of the convergents of $\alpha$. The following theorem due to Hurwitz is proved using properties of convergents of continued fraction expansions.

**Theorem 2.3.** Let $\alpha$ be an irrational number. Then there exist infinitely many rational numbers $\frac{p}{q}$ such that

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}.$$  

Up to here the approximability properties are seen to be shared by all irrationals alike. However when we look for sharper relations this no longer holds. It turns out that for $\alpha = \frac{\sqrt{5} - 1}{2}$ analogous statement does not hold if $\frac{1}{\sqrt{5}}$ replaced by a smaller constant. Same is the case for numbers equivalent to it, namely of the form $\frac{m\alpha + n}{pq + m}$, with $m, n, p$ and $q$ integers such that $mq - np = \pm 1$. However, but for these exceptions, for all other $\alpha$ the constant can be improved to $\frac{1}{\sqrt{8}}$. Then there is another number $\alpha$ such that the constant can not be improved any further for the numbers equivalent to it. Curiously the pattern repeats, and the constant can be improved to $\frac{5}{\sqrt{221}}$ for all others, and then again to $\frac{13}{\sqrt{1517}}$ for all but one more set of equivalent numbers, and so on; see [2] for details. The sequence $\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \frac{5}{\sqrt{221}}, \frac{13}{\sqrt{1517}}, \ldots$ continues in analogous fashion, and tends to $\frac{1}{3}$. The numbers in the sequence are called *Markov numbers* after A. Markov who discovered the phenomenon, in 1879. It may be noted that in view of the above, in particular, given $c > \frac{1}{3}$, for all but countably many irrationals $\alpha$, there are infinitely many rationals $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}| < \frac{c}{q^2}$. It turns out that $\frac{1}{3}$ is the smallest value for which this holds.

There is considerably more general theory on this topic; see for instance the recent paper [1] and the references suggested there. I will however end the general discussion here and move on to describing some extreme cases of interest.

### 3 Numbers with exceptional approximability features

A real (or complex) number $\alpha$ is said to be *algebraic* if it satisfies a polynomial equation of the form $r_0\alpha^d + r_1\alpha^{d-1} + \cdots + r_{d-1}\alpha + r_d = 0$, where $r_0, \ldots, r_d$
are rational numbers, not all 0; when \( \alpha \) is algebraic the smallest possible \( d \) for which there is such an equation satisfied by \( \alpha \) is called the degree of \( \alpha \). A number which is not algebraic is called transcendental.

In 1844 Liouville proved the following theorem.

**Theorem 3.1.** Let \( \alpha \) be an algebraic irrational number of degree \( d \). Then there exists \( c > 0 \) such that

\[
|\alpha - \frac{p}{q}| > \frac{c}{q^d}
\]

for all rationals \( \frac{p}{q} \).

As a consequence it follows that if \( \alpha \) is a number such that for every \( d \geq 1 \) there exists a rational \( \frac{p}{q} \) such that \( |\alpha - \frac{p}{q}| < \frac{1}{q^d} \), then \( \alpha \) is transcendental. It is quite easy to produce \( \alpha \) with this property; consider for example the sum of the convergent series \( \sum_{k=1}^{\infty} 2^{-k} \). Numbers with the above property are called Liouville numbers. It can be seen that they provide an uncountable collection of transcendental numbers.

Notwithstanding the fact that there are uncountably many numbers satisfying it, the condition for Liouville numbers is viewed as very strong. A following less demanding condition has attracted much attention in literature. An irrational number is said to be very well approximable (VWA for short) if there exist \( \epsilon > 0 \) and infinitely many rationals \( \frac{p}{q} \) such that

\[
|\alpha - \frac{p}{q}| < \frac{1}{q^{2+\epsilon}}.
\]

It turns out that even this degree of approximability is “rare” for real numbers. One way this is manifest is that the Lebesgue measure of the set of VWA numbers is 0. This means that a “randomly picked” real number (e.g. a number whose binary expansion is a random binary sequence) would not be VWA.

Some questions involving higher dimensional analogue of this condition have been the subject of some recent research of D. Kleinbock and G.A. Margulis, and it may be worthwhile to recall one of their results here. A \( n \)-tuple \((\alpha_1, \ldots, \alpha_n)\) is said to be VWA if there exist \( \epsilon > 0 \), infinitely many \( n \)-tuples of integers \((q_1, \ldots, q_n)\), and suitable integers \( p \) such that

\[
|q_1\alpha_1 + q_2\alpha_2 + \cdots + q_n\alpha_n - p| < \frac{1}{\|(q_1, \ldots, q_n)\|^{n(1+\epsilon)}}.
\]

(Here \( \|(q_1, \ldots, q_n)\| \) stands for “norm” which may be taken as \( |q_1| + \cdots + |q_n| \).) For \( n = 1 \) this coincides with the notion of VWA number as above. The
work of Kleinbock and Margulis[10], involving some very modern techniques, shows in particular the following: Let \( n \geq 1 \) and \( f_1, \ldots, f_n \) be polynomials in one variable, say \( t \), such that for any real numbers \( a_0, a_1, \ldots, a_n \), not all 0, \( a_0 + a_1 f_1 + \cdots + a_n f_n \) is a nonzero polynomial (i.e. 1, \( f_1, \ldots, f_n \) are linearly independent polynomials). Then for “almost all” \( t \) the \( n \)-tuple \( (f_1(t), \ldots, f_n(t)) \) is not VWA. The work established a longstanding conjecture of Sprindzuk, which in turn was inspired by an earlier conjecture of Mahler, from 1932, in the case \( f_i(t) = t^i \) as above.

We now come to a kind of behaviour which is at the other extreme. An irrational number \( \alpha \) is said to be badly approximable if there exists \( \delta > 0 \) such that for all rational numbers \( \frac{p}{q} \) we have

\[
|\alpha - \frac{p}{q}| > \frac{\delta}{q^2}.
\]

It may be noticed that by Liouville’s theorem recalled above quadratic irrationals, namely irrational numbers which satisfy a quadratic equation, are badly approximable. Interestingly, badly approximable numbers can be completely characterised in terms of continued fractions:

**Theorem 3.2.** An irrational number \( \alpha = [m_0, m_1, \ldots, m_k, \ldots] \) is badly approximable if and only if \( m_k \)'s are bounded, namely there exists \( M \) such that \( m_k \leq M \) for all \( k \).

The quadratic irrationals which form a part of badly approximable numbers consist precisely of \( \alpha = [m_0, m_1, \ldots, m_k, \ldots] \) which are “eventually periodic”, namely there exist positive integers \( p \) and \( k_0 \) such that \( m_{k+p} = m_k \) for all \( k \geq k_0 \).

Like the VWA numbers the badly approximable numbers also form a set of Lebesgue measure 0. It was however proved by Jarnik that they nevertheless form a large collection in terms of “Hausdorff dimension.” I will not go into the definition of Hausdorff dimension here, but content myself to say that it is a nonnegative number, which need not be an integer, which captures a sense of how large a set (strictly speaking a metric space) is. Jarnik’s theorem asserts that the Hausdorff dimension of the set of badly approximable numbers is 1, same as that of the number line itself, even though in general for subsets of the interval the number can be smaller than one.

Another sense in which the set is large was introduced by W.M. Schmidt. It depends on an idea of a two-person game and it would be my pleasure to recall it here, together with some results on its significance.

Consider the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), for some \( n \geq 1 \), with the usual distance. (The game may be thought of on any complete metric
space as well.) \( A \) and \( B \) are two players, and they are respectively assigned two numbers \( \alpha \) and \( \beta \) (strictly) between 0 and 1. A sample procedure for the game proceeds as follows: \( B \) chooses a closed ball in \( \mathbb{R}^n \), say \( B_0 \), with positive radius, say \( r_0 \). Then \( A \) chooses a closed ball, say \( A_1 \), of radius \( r_1 = \alpha r_0 \), contained in \( B_0 \). Then it is the turn of \( B \) again and he is to choose a closed ball \( B_1 \) of radius \( \beta r_1 \) contained in \( A_1 \). The game continues in this way with \( A \) and \( B \) taking turns in making choices: after \( k \geq 1 \) iterates \( B \) would have chosen a closed ball \( B_k \) of radius \( \alpha^k \beta^k r_0 \) contained in \( A_{k-1} \) and then \( A \) will choose a closed ball of radius \( \alpha^{k+1} \beta^k r_0 \) contained in \( B_k \). As the radii of \( \{A_k\} \) and \( \{B_k\} \) are decreasing to 0 and the balls are contained within each other as above, it follows that there is a unique point of intersection, viz. \( \cap_1^\infty A_k = \cap_1^\infty B_k = \{v\} \), with \( v \in \mathbb{R}^n \). A subset \( S \) of \( \mathbb{R}^n \) is preassigned, and the player \( A \) will be considered the winner if \( v \in S \) and \( B \) will be considered the winner if \( v \notin S \) (notice though that it involves infinitely many steps, unlike in a practical game, but the idea of winning or losing makes sense). Given the objective that the common point of intersection should be in \( S \), \( A \) will try to choose the balls \( A_k \), during his turns, so as to ensure that. On the other hand \( B \) will try to choose the balls \( B_k \) during his turns to avoid that happening. Now the question that concerns us is whether, given the set \( S \), \( A \) has a “winning strategy”, namely a way to choose the balls \( A_k \) during his turns, following the procedure as above, in such a way so that no matter what balls \( B \) chooses during his turns (within the rules of the game) the point of intersection will be in \( S \) (so as to be the winner). Notice that whether this is possible for the given \( S \) may also depend on the given \( \alpha \) and \( \beta \). If for \( S \) there is a winning strategy for \( A \) for given values \( \alpha, \beta \) we say that \( S \) is an \((\alpha, \beta)-winning \) set (for \( A \) - we will consistently suppress this part in the discussion below). For certain \( \alpha, \beta \) there may be no proper subset of \( \mathbb{R}^n \) which is a winning set; indeed this is the case if \( 1 - 2\alpha + \alpha \beta \leq 0 \), as can be readily proved. In particular if \( \alpha > \frac{1}{2} \), then for sufficiently small \( \beta > 0 \) there is no proper subset which is an \((\alpha, \beta)-winning \) set.

It stands to reason that the winning sets (for any \( \alpha, \beta \)) have to be “large sets”. Firstly \( S \) has to be dense in \( \mathbb{R}^n \), since otherwise \( B \) can choose \( B_0 \) to be outside \( S \), in which case \( A \) can not win. Also it can be seen that \( S \) has to be uncountable since otherwise \( B \) can force the points of \( S \) out of the chosen balls one by one (in finitely many steps in each case). The “largeness” of the winning sets however goes well beyond these simple manifestations. Schmidt showed that the Hausdorff dimension of an \((\alpha, \beta)-winning \) set in \( \mathbb{R}^n \) is at least \( (c - n \log \beta) / |\log \alpha \beta| \), where \( c \) is a constant depending only on \( n \). Let us call a subset \( S \) a winning set if it is a \((\frac{1}{2}, \beta)-winning \) set for all \( \beta > 0 \) (and hence \((\alpha, \beta)-winning \) set for all \( 0 < \alpha \leq \beta \) and \( \beta > 0 \)). The Hausdorff dimension
of any winning set is $n$, the maximum possible for a subset of $\mathbb{R}^n$. There is another curious fact about the winning sets which reflects their largeness: intersection of any two winning set is also a winning set, and more strongly intersection of any sequence of winning sets is also a winning set.

Schmidt proved that the set of badly approximable numbers is a winning set in real numbers [12]. Let me now conclude with a generalisation of this that I proved in higher dimensions; see [4] and [5]; the result was inspired by certain questions in dynamics of certain flows and the geometry of certain manifolds of negative curvature, which however are beyond the scope of this article.

**Theorem 3.3.** Let $\{v_k\}$ be a sequence of points in $\mathbb{R}^n$, the $n$-dimensional Euclidean space. Let $\{r_k\}$ be a sequence of positive numbers such that for any distinct $k$ and $l$ the distance between $v_k$ and $v_l$ is at least $\sqrt{r_k r_l}$. Let $S$ be the set of points $v$ such that for some $\delta > 0$, $v$ not contained in $B(v_k, \delta r_k)$, viz. the ball of radius $\delta r_k$ with centre at $v_k$, for any $k$. Then $S$ is a winning set in $\mathbb{R}^n$.

When $n = 1$, $\{v_k\}$ is a sequence enumerating the rationals, and $r_k = \frac{1}{q^2}$ if $v_k = \frac{p}{q}$, then the set as in the theorem consists precisely of badly approximable numbers.

**References**


S.G. Dani
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba
Mumbai 400 005
India
E-mail: dani@math.tifr.res.in