On invariant measures of the Euclidean algorithm

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Abstract. We study the ergodic properties of the additive Euclidean algorithm \( f \) defined in \( \mathbb{R}^2_+ \). A natural extension of \( f \) is obtained using the action of \( SL(2, \mathbb{Z}) \) on a subset of \( SL(2, \mathbb{R}) \). We prove that while \( f \) is an ergodic transformation with an infinite invariant measure equivalent to the Lebesgue measure, the invariant measure is not unique up to scalar multiples, and in fact there is a continuous family of such measures.

1. Introduction.

Let \( \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\} \). The Euclidean algorithm is the map defined by

\[
 f : (x_1, x_2) \in \mathbb{R}^2_+ \mapsto \begin{cases} (x_1 - x_2, x_2), & \text{if } x_1 \geq x_2 \\ (x_1, x_2 - x_1), & \text{otherwise}. \end{cases} \tag{1.1}
\]

When \( x_1 \) and \( x_2 \) are natural numbers the action of successive powers of \( f \) on \( (x_1, x_2) \) corresponds to the application of the Euclidean algorithm for finding the greatest common divisor (g.c.d.), say \( d \), of \( x_1 \) and \( x_2 \), and there exists a \( k \in \mathbb{N} \) such that \( f^k(x_1, x_2) = (d, 0) \) or \( (0, d) \). That is the source of the name for the transformation as above.

Let \( E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \), the two \( 2 \times 2 \) elementary matrices. They generate the group \( SL(2, \mathbb{Z}) \) consisting of all integral unimodular \( 2 \times 2 \) matrices. The map \( f \) as above is then given by

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2_+ \mapsto \begin{cases} E_1^{-1}x, & \text{if } x_1 \geq x_2 \\ E_2^{-1}x, & \text{otherwise}, \end{cases}
\]

where the matrices act on \( x \) as linear operators.

Let \( l \) denote the Lebesgue measure on \( \mathbb{R}^2_+ \). It is easy to see that \( f \) is a noninvertible map which is nonsingular with respect to \( l \); that is, if \( l(E) = 0 \) then \( l(f^{-1}(E)) = 0 \). By [N], \( f \) has the following property.
Theorem 1.1. For all $x \in \mathbb{R}_+^2$, the orbit of $x$ under $f$ equals the orbit of $x$ under the linear action of $SL(2, \mathbb{Z})$ graphed in $\mathbb{R}_+^2$, that is,

$$\bigcup_{n=0}^{\infty} \bigcup_{k=0}^{\infty} f^{-k}(\{f^n(x)\}) = \mathbb{R}_+^2 \cap SL(2, \mathbb{Z})x.$$ 

By a well-known result of Hedlund the linear action of $SL(2, \mathbb{Z})$ on $\mathbb{R}^2$ is ergodic; (see [BM] for instance). Theorem 1.1 therefore implies the following.

Corollary 1.2. The map $f$ is ergodic relative to the Lebesgue measure $l$.

This note was inspired by the question whether $f$ admits an invariant measure absolutely continuous with respect to $l$, and if it exists, such a measure is unique up to scalar multiples; we note that $f$ is not conservative and therefore, even though $f$ is ergodic, existence of an invariant density does not ensure it being unique. We prove the following.

Theorem 1.3 There exists a family $\{\nu_t\}_{t \in \mathbb{R}}$ of measures on $\mathbb{R}_+^2$ such that each $\nu_t$ is $f$-invariant and absolutely continuous with respect to $l$, and for $s, t \in \mathbb{R}$, $\nu_s$ and $\nu_t$ are not scalar multiples of each other, unless $s = t$.

It may be recalled that for a conservative ergodic nonsingular transformation an invariant measure (finite or infinite) equivalent to the original measure is unique up to scalar multiples, when it exists. In the light of Theorem 1.3 the Euclidean algorithm transformation as above furnishes a simple and natural example of a dissipative transformation for which this is not the case. Occurrence of such a phenomenon has also been noted earlier in respect of certain transformations associated with Engel series and Engel continued fractions (see [T], [S], p.78, and [HKS]); the authors are thankful to F. Schweiger for pointing this out.

Motivated by our paper, in [AM] it is proved that a dissipative, ergodic measure preserving transformation of a sigma-finite, non-atomic measure space always has many non-proportional, absolutely continuous invariant measures.

2. An invariant measure of $f$.

Consider the map $F$ of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ defined by

$$F : (x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \mapsto \begin{cases} (f(x), E_2y) = (E_1^{-1}, E_2y), & \text{if } x_1 \geq x_2, \\ (f(x), E_1y) = (E_2^{-1}, E_1y), & \text{otherwise}. \end{cases} \quad (2.1)$$
We see in particular that the Lebesgue measure on $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ (viewed canonically as $\mathbb{R}_+^4$) is invariant under the action of $F$. Using this we first describe a simple construction of an $f$-invariant measure absolutely continuous with respect to $l$. Let $\pi : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+^2$ be the canonical projection $\pi(x,y) = x$. Then $\pi(F(x,y)) = f(\pi(x,y))$ for all $(x,y)$. The recipe to obtain an $f$-invariant measure is to consider a suitable subset of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ invariant under $F$, and to integrate it along the fibers of $\pi$, (with respect to the other variable $y$); the set needs to be chosen so that the integrals along the fibers are finite.

For $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$, with $x = (x_1,x_2)$ and $y = (y_1,y_2)$, let $<x,y> = x_1y_1 + x_2y_2$ be the canonical scalar product in $\mathbb{R}^2$. We have

$$
\phi(x,y) = <x,y> = <E^{-1}x, E'y> = \phi(F(x,y)),
$$

where $E$ is either $E_1$ or $E_2$ and $E^t$ is the transpose of $E$. Thus $\phi$ is a nonconstant function invariant under $F$; (in particular, $F$ is not ergodic). Let

$$
\Omega = \{(x,y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : <x,y> \leq 1\}.
$$

Then $\Omega$ is $F$-invariant. Let $x = (x_1,x_2) \in \mathbb{R}_+^2$. Then for $y \in \mathbb{R}_+^2$ we see that $(x,y) \in \Omega$ if and only if $y$ belongs to the set

$$
\Omega(x) = \{z \in \mathbb{R}_+^2 : <x,z> \leq 1\}.
$$

The latter is a right-angled triangle whose catets are $1/x_1$ and $1/x_2$. Integrating the restriction of the Lebesgue measure to $\Omega$ along the fibers of $\pi$, as indicated above, we conclude that.

**Theorem 2.1.** The measure $d\nu = \frac{1}{2x_1x_2}dx_1dx_2$ is invariant under $f$.

3. A natural extension of $f$.

Let $F$ and $\Omega$ be as before; see (2.1) and (2.2). Let

$$
\Omega_1 = \{(x,y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : <x,y> = 1\}.
$$

Then $\Omega_1$ is $F$-invariant. Let $F_1$ be the restriction of $F$ to $\Omega_1$. Let $\rho : \Omega \to \Omega_1$ be the map defined by $(x,y) \in \Omega \to (x,y/<x,y>) \in \Omega_1$. Then it is easy to see that $F_1 \circ \rho = \rho \circ F$. We define a measure $\mu$ on $\Omega_1$ by setting, for any Borel subset $B \subset \Omega_1$,

$$
\mu(B) = \lambda(\rho^{-1}(B)).
$$
It can be seen that $\mu$ is a $\sigma$-finite measure on $\Omega_1$. Also the relation $F_1 \circ \rho = \rho \circ F$ shows that $\mu$ is invariant under $F_1$.

Clearly $F_1$ is an extension of $f$, with the projection $\pi_1$ from $(x, y) \in \Omega_1$ to the first coordinate $x$ as the extension map. We note that it is in fact a natural extension (see Aaronson [A], p.90 ff), in the sense that it is a minimal invertible extension.

**Theorem 3.1.** The map $F_1$ is a natural extension of $f$.

**Proof.** Since $F_1$ is an extension, with an infinite invariant measure $\mu$ as above, to prove the theorem it suffices to show that the partition of $\Omega_1$ into equivalence classes of the relation defined by $(x, y) \sim (x', y')$, for $(x, y), (x', y') \in \Omega_1$, if $\pi_1(F_1^{-1}((x, y))) = \pi_1(F_1^{-1}((x', y')))$ for all $i \geq 0$, is the trivial partition mod $\mu$. Let $(x, y), (x', y') \in \Omega_1$ and let $F_1^{-1}((x, y)) = (x_i, y_i)$ and $F_1^{-1}((x', y')) = (x_i', y_i')$ for all $i \geq 0$. Then $f(x_i) = x_{i-1}$ for all $i \geq 1$. By the definition of $f$ there exist $A_i \in \{E_1^{-1}, E_2^{-1}\}$ such that $f(x_i) = A_i x_i = x_{i-1}$ for all $i \geq 1$. Then $F_1((x_i, y_i)) = (A_i x_i, A_i^{-1} y_i)$ for all $i \geq 1$. Hence $y_i = A_i^i y_{i-1}$ for all $i \geq 1$. Similarly $y_i' = A_i^i y_{i-1}'$ for all $i \geq 1$. But there exists a unique number $y$ such that the sequence defined by $y_0 = y$ and $y_i = A_i^i y_{i-1}$ for all $i \geq 1$ consists only of nonnegative numbers. This shows that $y' = y$, and therefore the partition as above is the trivial partition.

We now give another realisation of the natural extension. For this we identify the subset $\Omega_1$ canonically with the subset of $SL(2, \mathbb{R})$ given by

$$\Omega^{(1)} = \left\{ \begin{pmatrix} x_1 & -y_2 \\ x_2 & y_1 \end{pmatrix} \in SL(2, \mathbb{R}) : x_1, x_2, y_1, y_2 \geq 0 \right\}. \quad (3.1)$$

The map $F_1$ then corresponds to

$$F_1 : g \in \Omega^{(1)} \mapsto E^{-1} g \in \Omega^{(1)}, \quad (3.2)$$

where if $g = \begin{pmatrix} x_1 & -y_2 \\ x_2 & y_1 \end{pmatrix}$ then $E = E_1$ if $x_1 \geq x_2$, and $E = E_2$ otherwise.

Using Theorem 1.1 and (3.1 - 3.2), we deduce the following.

**Theorem 3.2.** Let $g = \begin{pmatrix} x_1 & -y_2 \\ x_2 & y_1 \end{pmatrix} \in \Omega^{(1)}$. Then the orbit of $g$ under $F_1$ equals the orbit of $g$ under the action of $SL(2, \mathbb{Z})$ on $SL(2, \mathbb{R})$ by translations on the left, graphed in $\Omega^{(1)}$, that is,

$$\bigcup_{n=-\infty}^{\infty} F_1^n \{ g \} = \Omega^{(1)} \cap SL(2, \mathbb{Z}) g.$$
Proof. Let \( v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). If \( x_1/x_2 \) is rational then the assertion follows easily from the Euclidean algorithm for pairs of natural numbers. Now suppose that \( x_1/x_2 \) is irrational. Let \( A \in SL(2, \mathbb{Z}) \) be such that \( Ag \in \Omega^{(1)} \). In order to prove our claim it suffices to show that there exists \( k \in \mathbb{Z} \) such that \( Ag = F_1^k(g) \). As \( Ag \in \Omega^{(1)} \), \( Av \in \mathbb{R}_+^2 \). By Theorem 1.1, there exist \( m, n \geq 1 \) such that \( f^m(Av) = f^n(v) \). By the definition of \( f \), there exist \( \gamma, \gamma' \) in the semigroup generated by \( E_1^{-1} \) and \( E_2^{-1} \), such that \( f^n(v) = \gamma v \) and \( f^m(Av) = \gamma' Av \). Thus we get \( \gamma' Av = \gamma v \), which means that \( \gamma^{-1} \gamma' A \) fixes \( v \). Since \( x_1/x_2 \) is irrational this implies that \( \gamma^{-1} \gamma' A \) is the identity matrix, and so \( \gamma' A = \gamma \). Since \( f^n(v) = \gamma v \) and \( f^m(Av) = \gamma' Av \), comparing the definitions of \( F_1 \) and \( f \) we see that \( F_1^m(Ag) = \gamma' Ag = \gamma g = F_1^n(g) \). Therefore \( Ag = F_1^{m-n}(g) \). This proves our claim.

4. Invariant densities for \( f \).

Using the model for the natural extension as described in the last section we now provide a construction of a large class of measures on \( \mathbb{R}_+^2 \) which are absolutely continuous with respect to the Lebesgue measure \( l \), and \( f \)-invariant.

For simplicity of notation let \( G = SL(2, \mathbb{R}) \) and \( \Gamma = SL(2, \mathbb{Z}) \). The quotient space \( \Gamma \backslash G \) can be realised canonically as the space of unimodular lattices in \( \mathbb{R}^2 \), associating to each (right) coset \( g, g' \in G \), the lattice in \( \mathbb{R}^2 \) generated by the rows of \( g \) (it being independent of the representative \( g \) in \( \Gamma g \)). The space \( \Gamma \backslash G \) carries a unique probability measure, say \( m \), invariant under the action of \( G \) on \( \Gamma \backslash G \) on the right (see [BM]).

For each \( s \in \mathbb{R} \) let \( g_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \) and for every \( t \in \mathbb{R} \) let \( h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \). Then \( \{g_s\} \) and \( \{h_t\} \) are one-parameter subgroups of \( G \). Their actions on \( \Gamma \backslash G \) induced by the \( G \)-action correspond, respectively, to the geodesic and horocycle flows associated with the modular surface, and these are both ergodic with respect to the measure \( m \).

Let \( L^\infty(\Gamma \backslash G)^+ \) denote the space of all nonnegative bounded measurable functions on \( \Gamma \backslash G \). We now associate to each \( \varphi \in L^\infty(\Gamma \backslash G)^+ \) a \( f \)-invariant absolutely continuous measure on \( \mathbb{R}_+^2 \). Let \( \lambda \) denote the Haar measure on \( G \). For any \( \varphi \in L^\infty(\Gamma \backslash G)^+ \) let \( \mu_\varphi \) be the measure on \( \Omega^{(1)} \) defined by

\[
\mu_\varphi(A) = \int_A \varphi(\Gamma g) d\lambda(g),
\]
for all Borel subsets $A$ of $\Omega^{(1)}$. We claim that $\mu_\varphi$ is a $F_1$-invariant measure on $\Omega^{(1)}$. In view of Theorem 3.1 every Borel subset $A$ of $\Omega^{(1)}$ can be decomposed as a countable disjoint union $A = \cup A_i$ such that on each $A_i$ the action of $F_1$ coincides with the action of an element $\gamma_i$ of $\Gamma$. Therefore to prove the claim it suffices to see that $\mu_\varphi(\gamma A) = \mu_\varphi(A)$ for all Borel subsets $A$ of $\Omega^{(1)}$ and $\gamma \in \Gamma$ such that $\gamma A$ is contained in $\Omega^{(1)}$; this is clear from the definition of $\mu_\varphi$, and proves the claim.

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}_+^2$. The action of $G = SL(2, \mathbb{R})$ on $\mathbb{R}^2 - (0)$ is transitive, and thus $\mathbb{R}^2 - (0)$ may be realised as a homogeneous space $G/U$, with $U = \{ h_t : t \in \mathbb{R} \}$, the stabiliser of $e_1$. By the Fubini theorem for homogeneous spaces, for any $\varphi \in L^\infty(\Gamma \backslash G)$ we have

$$\int \psi(g) d\lambda(g) = \int_{\mathbb{R}^2 - (0)} \left( \int_{\mathbb{R}} \psi(gh_t) dt \right) dl,$$

where $l$ is the Lebesgue measure on $\mathbb{R}^2$, and the expression in parenthesis is viewed as a function on $\mathbb{R}$ with $\int_U \psi(gh_t) dt$ as the value at the point $ge_1$, namely at $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ if $g = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$. Hence for any bounded measurable function $\psi$ vanishing outside $\mathbb{R}^2$ with $\int \psi(gh_t) dt$ we have

$$\int \psi(g) d\mu_\varphi(g) = \int \psi(g) \varphi(\Gamma g) d\lambda(g) = \int_{\mathbb{R}^2 - (0)} \left( \int_{\mathbb{R}} \varphi(\Gamma gh_t) \psi(gh_t) dt \right) dl,$$

where the parenthetical integral is the value of the outer integrand at the point $ge_1$. This implies that the image of $\mu_\varphi$ on $\mathbb{R}_+^2$ is a $\sigma$-finite measure, say $\nu_\varphi$, which is absolutely continuous with respect to the Lebesgue measure $l$, and $d\nu_\varphi = \varphi(x_1, x_2) dx_1 dx_2$, with

$$\varphi(x_1, x_2) = \int_{\mathbb{R}} \varphi(\Gamma gh_t) \chi(gh_t) dt,$$

where $\chi$ denotes the characteristic function of $\Omega^{(1)}$ in $G$, and $g \in G$ is any element such that $ge_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Thus we have for each $\varphi \in L^\infty(\Gamma \backslash G)^+$ a measure $\nu_\varphi$ on $\mathbb{R}_+^2$ which is absolutely continuous with respect to $l$. Also, since $\mu_\varphi$ is $F_1$-invariant it follows that $\nu_\varphi$ is $f$-invariant.

Now let $(x_1, x_2) \in \mathbb{R}_+^2$ and choose $g = \begin{pmatrix} x_1 & -x_2^{-1} \\ x_2 & 0 \end{pmatrix}$. Then $gh_t = \begin{pmatrix} x_1 & x_1 t - x_2^{-1} \\ x_2 & x_2 t \end{pmatrix} \in \Omega^{(1)}$ if and only if $x_1 t - x_2^{-1} < 0$ and $x_2 t > 0$, or
equivalently if and only if \( 0 < t < 1/x_1 x_2 \). Therefore we get that

\[
\varphi(x_1, x_2) = \int_0^{1/x_1 x_2} \varphi(\Gamma \left( \begin{array}{cc} x_1 & -x_2^{-1} \\ x_2 & 0 \end{array} \right) t) dt \\
= \int_0^{1/x_1 x_2} \varphi(\Gamma \left( \begin{array}{cc} x_1 & x_1 t - x_2^{-1} \\ x_2 & x_2 t \end{array} \right) ) dt. \tag{4.2}
\]

We note that when \( \varphi \) is chosen to be the constant function 1 the measure \( \nu_1 = \nu_\varphi \) is \( \frac{1}{x_1 x_2} dx_1 dx_2 \), the invariant measure as in Theorem 2.1. It may also be noted that for any \( \varphi \in L^\infty(\Gamma \backslash G) \) the measure \( \nu_\varphi \) is absolutely continuous with respect to \( \nu_1 \) and the Radon-Nikodym derivative \( \frac{d\nu_\varphi}{d\nu_1} \) is bounded by the essential supremum of \( \varphi \).

**Proposition 4.1** Let \( \varphi \) be the characteristic function of a nonempty open subset \( \Theta \) of \( \Gamma \backslash G \). Then \( \frac{d\nu_\varphi}{d\nu_1} \) takes the value 1 on a set of positive Lebesgue measure.

**Proof.** We note that \( \Gamma \Omega^{(1)} = G \); this may be deduced by observing that every lattice of row vectors has a basis of the form \( \{(a, -b), (c, d)\} \), with \( a, b, c, d > 0 \). Let \( \Psi \) be a compact subset of \( \Omega^{(1)} \) with nonempty interior, such that \( \Gamma \backslash \Gamma \Psi \) is contained in \( \Theta \) and \( \Psi e_1 = \{g e_1 : g \in \Psi\} \) is contained in \( \{(x_1, x_2) \in \mathbb{R} : x_1, x_2 > 0\} \). Then there exists a \( \delta > 0 \) such that \( \Gamma g h_t \in \Theta \) for all \( g \in \Psi \) and \( t \in (-\delta, \delta) \). Also there exists a \( C > 0 \) such that if \( g \in \Psi \) and \( g e_1 = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 \), then \( \frac{1}{x_1 x_2} \leq C \). Now consider any element \( g' = gg_s \), with \( g \in \Psi \) and \( s > \log \frac{1}{2} (C/\delta) \), such that \( \Gamma g' \in \Gamma \Psi \). Then we have, for all \( t \in \mathbb{R} \),

\[
\varphi(\Gamma g' h_t) \chi(g' h_t) = \varphi(\Gamma g' h_t) \chi(g g_s h_t) = \varphi(\Gamma g' h_t) \chi(g g_s h_t g' - s),
\]

where (as before) \( \chi \) is the characteristic function of \( \Omega^{(1)} \), and the last equality holds since \( \Omega^{(1)} \) is invariant under the right translations by \( \{g_s\} \). As \( g_s h_t g' - s = h_{e_2}, t \) and \( e^{2s} t > C \) if \( |t| \geq \delta \), it follows that \( \chi(g g_s h_t g' - s) = 0 \) for all \( t \) such that \(|t| \geq \delta \). On the other hand, for \( t \in (-\delta, \delta) \), \( \varphi(\Gamma g' h_t) = 1 \), since \( \Gamma g' \in \Gamma \Psi \subset \Theta \). Thus we see that \( \varphi(\Gamma g' h_t) \chi(g' h_t) = \chi(g' h_t) \) for all \( t \in \mathbb{R} \). By (4.1) this shows that \( \frac{d\nu_\varphi}{d\nu_1}(g' e_1) = 1 \).

Since the flow induced by \( \{g_s\} \), namely the geodesic flow, is ergodic, the set of elements \( g' \) in \( \Omega^{(1)} \) for which the condition as above is satisfied is a set of positive (Haar) measure, and hence its image in \( \mathbb{R}^2_+ \) is a set of positive Lebesgue measure. This proves the proposition. \( \Box \)
Proof of Theorem 1.3. Let $S$ be a smooth open surface in $\Gamma \backslash G$, transversal to the horocycle flow, namely the action of $\{h_t\}$ on the right, such that $(\sigma,t) \mapsto \sigma h_t$ is a diffeomorphism of $S \times (-1,1)$ onto an open subset, say $B$, of $\Gamma \backslash G$. For each $r \in (-1,1)$ let $B_r$ be the image of $S \times (-r,r)$ in $\Gamma \backslash G$. Then each $B_r$ is an open subset of $\Gamma \backslash G$; let $\varphi_r$ be the characteristic function of $B_r$. To prove the theorem it suffices to show that no two of the $f$-invariant measures $\{\nu_{\varphi_r}\}_{r \in (-1,1)}$ are scalar multiples of each other. Since by Proposition 4.1 their essential suprema are 1, they can be scalar multiples of each other only if they are equal. We shall show that they are in fact distinct.

Now let $a, b \in (-1,1)$, say $-1 < a < b < 1$, and consider $\nu_{\varphi_a}$ and $\nu_{\varphi_b}$. Let $g \in \Omega^{(1)}$ be arbitrary. Then the set, say $T$, of $t$ in $\mathbb{R}$ for which $gh_t \in \Omega^{(1)}$ is an interval of length $1/x_1x_2$, where $x_1, x_2$ are such that $ge = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Let $T_1 = \{t \in T : \Gamma gh_t \in B\}$, $T_a = \{t \in T : \Gamma gh_t \in B_a\}$ and $T_b = \{t \in T : \Gamma gh_t \in B_b\}$. We note that $T_1$ is a string of disjoint intervals, say $k$ of them, and the lengths of all except possibly the first and the last one are 2. Suppose $k \geq 3$. Then each of the middle intervals intersects $T_a$ and $T_b$ in intervals of lengths $2a$ and $2b$ respectively; the end intervals could be smaller, but the length of the intersection with $T_b$ is at least as much as the length of the intersection with $T_a$. Thus the Lebesgue measures of $T_a$ and $T_b$ are $2(k-2)a + c$ and $2(k-2)b + d$ respectively, where $c$ and $d$ are the total lengths of the segments in the first and the last intervals, and we have $0 \leq c \leq d \leq 4$. Hence

$$\frac{d\nu_{\varphi_a}}{d\nu_1}(ge) = \int_{\mathbb{R}} \varphi_a(\Gamma gh_t)\chi(gh_t)dt < \int_{\mathbb{R}} \varphi_b(\Gamma gh_t)\chi(gh_t)dt = \frac{d\nu_{\varphi_b}}{d\nu_1}(ge),$$

provided $k$ as above is at least 3.

To complete the proof it suffices therefore to show that the set of $g$ in $\Omega^{(1)}$ for which $k = k(g)$ as above is at least 3 has positive measure. We shall show that for any given $k_0 \in \mathbb{N}$ the set of $g$ in $\Omega^{(1)}$ for which $k(g) \geq k_0$ is a set of positive measure.

Let $k_0 \in \mathbb{N}$ be given. Let $\varphi$ be the characteristic function of $B$, and let $\theta$ be the function on $\Gamma \backslash G$ defined by $\theta(\Gamma g) = \int_0^1 \varphi(\Gamma gh_t)dt$. The action of $h := h_1$ on $\Gamma \backslash G$ is ergodic (see [BM]), and hence

$$\frac{1}{n}\sum_{i=0}^{n-1}\theta(\Gamma gh^i) \longrightarrow \int_{\Gamma \backslash G} \theta dm = m(B), \ a.e..$$

Hence there exist $n \geq 4k_0/m(B)$ and a Borel subset $E$ of $\Gamma \backslash G$ such that $m(E) > 0$ and

$$\frac{1}{n}\sum_{i=0}^{n-1}\theta(\Gamma gh^i) \geq m(B)/2.$$
for all \( g \) in \( G \) such that \( \Gamma g \in E \). Then for any \( g \) with \( \Gamma g \in E \) we have
\[
\int_0^m \varphi(\Gamma gh_t) \, dt = \sum_{i=0}^{m-1} \theta(\Gamma gh^i) \geq \frac{n}{2} m(B) \geq 2k_0.
\]

Now let \( \Psi \) be a compact subset of \( \Omega^{(1)} \) with nonempty interior, contained in the interior of \( \Omega^{(1)} \), and such that \( \Gamma \backslash \Gamma \Psi \) is contained in \( B \). Then there exists a \( \delta > 0 \) such that \( \Psi h_t \) is contained in \( \Omega^{(1)} \) for all \( t \in (-\delta, \delta) \). Since the geodesic flow is ergodic there exists \( s > \frac{1}{2} \log n/\delta \) such that \( m(E \cap (\Gamma \backslash \Gamma \Psi g_s)) > 0 \). Now consider any \( g \) in \( G \) such that \( \Gamma g \in E \) and \( g = g' g_{-s} \) for some \( g' \in \Psi \). We note that \( gh_t = g' g_{-s} h_t \in \Omega^{(1)} \) if and only if \( g' h_{e^{-2s}t} = g' g_{-s} h_t g_s \in \Omega^{(1)} \), and since \( g' \in \Psi \) the set of \( t \) for which this holds contains the interval \((-\delta e^{2s}, \delta e^{2s})\), which in turn contains \((0, n)\). Also, since \( \Gamma g \in E \), we have \( \int_0^n \chi_B(\Gamma gh_t) \, dt \geq 2k_0 \). Thus for such a \( g \) the set \( T \) as in the preceding argument contains the interval \((0, n)\), and the subset \( T_1 \cap (0, n) \) has Lebesgue measure at least \( k_0 \). Since \( T_1 \) is a union of intervals of length at most \( 2 \), we get that \( T_1 \) contains at least \( k_0 \) intervals. This completes the argument and the proof of the theorem.

5. Interval exchange transformations.

We conclude with some remarks setting Theorem 1.3 in a broader context. Let \( \lambda_1, \lambda_2 \) be positive. Set \( I = [0, \lambda_1 + \lambda_2], I_1 = [0, \lambda_1] \) and \( I_2 = [\lambda_1, \lambda_1 + \lambda_2] \). The map \( T : I \to I \) defined by \( Tx = x + \lambda_2 \), if \( x \in I_1 \), and \( Tx = x - \lambda_1 \), if \( x \in I_2 \), is an interval exchange of the intervals \( I_1 \) and \( I_2 \). The map \( f \) as in (1.1) corresponds to Rauzy induction defined for interval exchange transformations (see [V1]) in the special case with two intervals. In the light of our proof of Theorem 1.3 and the natural extension \( \mathcal{U} \) defined by Veech (see [V1], p. 219), it may be seen that a result analogous to Theorem 1.3 would hold for the Rauzy induction \( \mathcal{I} \) (see [V2], p. 1390).

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