WEIGHTLESS COHOMOLOGY OF ALGEBRAIC VARIETIES

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Abstract. Using Morel’s weight truncations in categories of mixed sheaves, we attach to varieties over finite fields or the complex numbers a series of groups called the weightless cohomology groups. These lie between the usual cohomology and intersection cohomology, have natural ring structures, and are functorial for certain morphisms.

1. Introduction

Let \( k \) be the field \( \mathbb{C} \) of complex numbers or a finite field. If \( k \) is finite let \( l \) be a prime different from the characteristic of \( k \). For a variety \( X/k \) we let \( H^*(X) \) denote the singular cohomology \( H^*(X, \mathbb{Q}) \) if \( k = \mathbb{C} \) or the étale cohomology \( H^*(X_\overline{k}, \mathbb{Q}_l) \) if \( k \) is finite. Similarly, \( IH^*(X), H^*_{BM}(X), \) etc. denote the appropriate intersection cohomology, Borel-Moore homology, etc. Let \( D^b(X) \) denote the derived category of mixed Hodge modules \([Sai90]\) in the case \( k = \mathbb{C} \) or the derived category of mixed \( l \)-adic complexes \([BBD82]\) if \( k \) is finite. \((\cdot)\) denotes the Tate twist.

1.1. Properties of weightless cohomology. Using truncation functors introduced by S. Morel \([Mor08]\), we attach to a variety \( X/k \) a complex \( EC_X \in D^b(X) \) which we call the weightless complex of \( X \). Its hypercohomology groups \( EH^*(X) := \mathbb{H}^*(X, EC_X) \) are the weightless cohomology groups of \( X \). These are finite-dimensional \( \mathbb{Q} \)-vector spaces (if \( k = \mathbb{C} \)) or \( \mathbb{Q}_l \)-vector spaces (if \( k \) is finite) with the following properties (Theorem 4.2.1):

(i) If \( X \) is of dimension \( d \) then \( EH^i(X) \neq 0 \) only if \( i \in [0, 2d] \). If \( \hat{X} \to X \) is the normalization then \( EH^*(\hat{X}) = EH^*(X) \). If \( X \) is smooth then \( EH^*(X) = H^*(X) \).

(ii) \( EH^*(X) \) is a graded-commutative \( \mathbb{Q} \)-algebra (if \( k = \mathbb{C} \)) or \( \mathbb{Q}_l \)-algebra (if \( k \) is finite).

(iii) \( EH^*(X \times Y) \cong EH^*(X) \otimes EH^*(Y) \).

(iv) \( EH^*(X) \) lives between the usual cohomology \( H^*(X) \), and the intersection cohomology \( IH^*(X) \), that is, there are homomorphisms

\[
H^*(X) \to EH^*(X) \to IH^*(X)
\]

factoring the natural homomorphism \( H^*(X) \to IH^*(X) \). The first morphism is a ring homomorphism and the second is an \( H^*(X) \)-module homomorphism. In general, these homomorphisms are neither injective nor surjective.

(v) If \( f : Y \to X \) is a morphism of varieties such that the image of each irreducible component of \( f \) meets the smooth locus of \( X \) then there is a natural ring homomorphism \( f^* : EH^*(X) \to EH^*(Y) \) compatible with \( H^*(X) \to H^*(Y) \). If \( f : Y \to X \) and \( g : Z \to Y \) are morphisms such that \( f, g, \) and \( f \circ g \) satisfy this condition (that the
image of each irreducible component of the domain meets the smooth locus of the codomain) then \((f \circ g)^* = g^* \circ f^*\).

(vi) If \(d = \dim X\) then \(EH^{2d}(X) = IH^{2d}(X)\) and \(EH^{2d-1}(X) \to IH^{2d-1}(X)\).

(vii) There is a module action of \(EH^*(X)\) on \(IH^*(X)\) factoring the action of \(H^*(X)\) on \(IH^*(X)\).

These statements reflect properties of the complex \(EC_X\) in \(D^b(X)\).

If \(k = \mathbb{C}\) then \(EH^*(X)\) carries a (rational) mixed Hodge structure. If \(k\) is finite then \(EH^*(X)\) has an action of \(\text{Gal}(\overline{k}/k)\) and hence a weight filtration coming from the action of Frobenius. The ring structure in (ii), the homomorphisms in (iii)–(vi), and the action in (vii) respect these structures. Furthermore, the weights of \(EH^*(X)\) satisfy the following constraints (Corollary 4.3.4):

(viii) \(Gr_j^W EH^i(X) \neq 0\) only if \(j \in [0, 2i] \cap [2i - 2 \dim X, 2 \dim X]\).

(ix) If \(X\) is proper then \(Gr_j^W EH^i(X) = 0\) for \(j > i\) and

\[
Gr_i^W EH^i(X) = \text{im}(EH^i(X) \to IH^i(X)).
\]

If \(X\) is smooth then \(Gr_j^W EH^i(X) = 0\) for \(j < i\).

Thus \(EH^*(X)\) looks very much like the cohomology of an algebraic variety.

Since \(EH^*(X)\) is defined via a complex \(EC_X\) in \(D^b(X)\), it has a compactly supported variant \(EH^*_c(X) = H^*_c(X, EC_X)\), and the natural map \(EH^*_c(X) \to EH^*(X)\) is an isomorphism when \(X\) is proper. There are also Poincaré dual homology theories defined by \(EH^{BM}_c(X) = H^{-i}(X, \mathbb{D}EC_X)\) and \(EH_c(X) = H^{-i}_c(X, \mathbb{D}EC_X)\) where \(\mathbb{D}\) is the Verdier duality functor on \(D^b(X)\) and \(d = \dim X\). The fundamental class homomorphism then factors as

\[
H^*(X) \to EH^*(X) \to IH^*(X) = IH^{BM}_{2d-\ast}(X)(-d) \to EH^{BM}_{2d-\ast}(X)(-d) \to IH^{BM}_{2d-\ast}(X)(-d)
\]

\(EH^{BM}_c(X)\) and \(EH_c(X)\) are graded modules for \(EH^*(X)\) and satisfy properties dual to those of \(EH^*(X)\) and \(EH^*_c(X)\) above.

1.2. Motivic versions. In the recent paper \text{[AZ12]} of Ayoub and Zucker, the authors construct, for any variety \(X\) over a field of characteristic zero, a ring object in a category of motivic sheaves on \(X\). In \text{[Vai13]} the second named author shows that the construction realizes to our \(EC_X\) under an appropriate realization functor to Saito’s mixed Hodge modules. Such a realization has recently been constructed in \text{[Ivo14]}. However the construction of \(EC_X\) here is relatively elementary and the additional properties (specifically, (iv), (vi), and (ix)) here are necessary for some applications in \text{[Nai10]}. \footnote{The construction of \(EC_X\) and the proofs of its properties were worked out by the authors in 2009, before the first version of \text{[AZ12]} became available, with a view to the applications to Shimura varieties in \text{[Nai10]}.}

In fact in \text{[Vai13]} an alternate construction of the motive of Ayoub and Zucker is given along the lines suggested by the construction of \(EC_X\) here and properties analogous to those above are established, in arbitrary characteristic (for the analogue of relation (iv), the motivic intersection complex as defined in \text{[Wil12]} or \text{[Wil12IC]} is used - it is known to exist in several cases of interest, for example, for all Shimura varieties). The methods in \text{[Vai13]} can also be used to recover several results here, even in absence of a realization functor. (In particular, this gives, using suitable \(l\)-adic realization functors, a definition of weightless cohomology for varieties satisfying (i)–(vii) over any field.)
1.3. The theory $EH^*(X)$ is, in general, closer to ordinary cohomology than to intersection cohomology. Indeed, $H^*(X)$ and $EH^*(X)$ tend to differ only for varieties with rather complicated singularities (we do not attempt to make this precise). For example, they coincide for toric varieties (in any characteristic, see Ex. 5.0.4) or for complex algebraic varieties with isolated rational singularities (by a result of [ABW11], cf. Ex. 5.0.6).

On the other hand, compactified Hilbert-Blumenthal varieties (of arbitrary dimension) give examples where $EC_X = IC_X$ and hence $EH^*(X)$ and $IH^*(X)$ coincide (and differ from $H^*(X)$ if $\dim X > 2$). Note that when $EH^*(X)$ and $IH^*(X)$ coincide then intersection cohomology has a natural ring structure; it would be interesting to see whether in other examples where $IH^*(X)$ is known (or expected) to have a ring structure the underlying explanation is that $EC_X = IC_X$.

An interesting example of the construction $EC_X$, and in fact the one that motivated the results here, is when $X$ is the Baily-Borel compactification of a locally symmetric variety or Shimura variety, which has log canonical singularities at the boundary. In this case $EH^*(X)$ is the cohomology of the group-theoretically defined reductive Borel-Serre compactification [Nai12], cf. also [AZ12], which plays an important role in the cohomology of arithmetic groups and automorphic forms. In [Nai10], the results of this paper are used together with arguments about automorphic forms to prove new results about the cohomology of the reductive Borel-Serre compactifications of locally symmetric varieties.

(These examples and others are discussed in §5.)

1.4. The weightless complex. The main technical tools in the construction of the weightless complex $EC_X$ and in the proof of its properties are the theory of weights and Morel’s truncation functors [Mor08] and the theory of $t$-structures and gluing [BBD82]. Given a stratification $X = \bigsqcup_i S_i$ of a variety $X$ and a function $a$ from the set of strata to $\mathbb{Z} \cup \{\pm \infty\}$, Morel defines a $t$-structure $(\underline{wD} \leq a, \underline{wD} > a)$ on $D^b(X)$, and hence truncation functors $w_{\leq a}, w_{> a}$ on $D^b(X)$. The complex $EC_X$ is defined using the truncation given by the function $\dim$ with $\dim(S_i) := \dim S_i$. Assume that the stratification is such that $U = S_0$ is smooth, open, and dense in $X$. Let $j : U \to X$ be the inclusion; assume further that $Rj_* \mathbb{Q}_U$ and its Verdier dual are constructible with respect to the stratification. (For example, if $k = \mathbb{C}$ this holds if the stratification is Whitney. Here $\mathbb{Q}_U$ means the constant l-adic sheaf if $k$ is finite and the analogous object if $k = \mathbb{C}$.) Then $EC_X$ is defined by

$$EC_X := w_{\leq \dim} Rj_* \mathbb{Q}_U.$$

We call $EC_X$ the weightless complex of $X$ for the following reason: If $x \in X$ is a closed point then the local stalk cohomology groups $H^i(EC_X)_x$ are of weight zero for all $i$. (In the Hodge-theoretic setting they are of type $(0,0)$ and in Galois settings of Artin type.) If $X$ is equidimensional of dimension $d$ and the intersection complex $IC_X = (j_* \mathbb{Q}_U[d])[-d]$ is constructible with respect to the stratification one also has

$$EC_X = w_{\leq \dim} IC_X$$

as a consequence of Morel’s remarkable formula $IC_X = w_{\leq d} Rj_* \mathbb{Q}_U$ where $w_{\leq d}$ is the truncation functor for the constant function $d = \dim X$. Finally, for a surjective birational morphism $\pi : Y \to X$ from a smooth variety we have a natural isomorphism

$$EC_X = w_{\leq \dim} R\pi_* \mathbb{Q}_Y.$$
(assuming that $R\pi_*\mathcal{Q}_Y$ and its Verdier dual are constructible for the stratification). \footnote{It follows from this formula (cf. Ex. \ref{ex:5.0.2}) that if $X$ has an isolated singularity at $x$ or more generally if the singular locus of $X$ is smooth and $\pi^*\mathcal{R}_j\mathcal{Q}_{\bar{U}}$ is lisse, where $j : U \hookrightarrow X$ is the inclusion of smooth locus and $i : X-U \hookrightarrow X$ the complement, then $H^*(EC_X)_s$ is simply the weight zero part of the cohomology of the fibre $H^*(\pi^{-1}(x))$. This is not true when the singularities of $X$ are more complicated, as Ex. \ref{ex:5.0.10} shows.}

This implies that the pullback map $H^*(X) \to H^*(Y)$ factors through $EH^*(X)$, suggesting that $EH^*(X)$ should be thought of as the cohomology of a genuine space which behaves like a “partial resolution” of $X$ such that all other resolutions of $X$ factor through it. Note, however, that even if $X$ does have such a resolution $Y \to X$ it need not be the case that $EH^*(X) = H^*(Y)$.

In the body of the paper we will only be interested in $\mathcal{g}$ which depend only on the dimensions of the strata, so that $\mathcal{g}$ is replaced by a function $D : \mathbb{Z}^{\geq 0} \to \mathbb{Z}$. Then $\dim$ corresponds to the identity function $\text{Id}$. We show that, much as in the case of $t$-structures associated with perversities, for certain choices of $D$ (monotone step functions, see \ref{sec:3.1.2}), one can define $t$-structures ($wD \leq D$, $wD > D$) on the appropriate derived category which are independent of the stratification (\ref{sec:3.1.3}). The associated truncation functors $w_{\leq D}, w_{> D}$ are then defined and the special case $D = \text{Id}$ gives the earlier construction of $EC_X$. The Verdier dual of $EC_X$, used to define the groups $EH^b_{BM}(X)$, is also given by a suitable truncation $w_{\leq D}$.

### 1.5. Generalizations

The definition of $EC_X$, its main properties, and their proofs generalize immediately to the situation with coefficients in a pure lisse $\mathbb{Q}$-sheaf (when $k$ is finite) or pure smooth Hodge module (equivalently, a polarizable variation of Hodge structure, when $k = \mathbb{C}$) on a smooth open dense subset of $X$. (Purity is essential for Morel’s truncation theory.) The construction also makes sense in any suitable theory of mixed sheaves (e.g. in the sense of Saito \cite{Sai06}), and the properties outlined above for $EH^*(X)$ will hold in any such theory, with the same proofs. In particular, a theory of mixed $l$-adic complexes is now available for varieties over a number field \cite{Mor11}; the weightless cohomology groups are then representations of the absolute Galois group.

Morel pointed out to us that the construction and properties of weightless cohomology in the $l$-adic setting work over any field which is finitely generated over the prime field, or a totally inseparable extension of such a field, with the same proofs. She also pointed out that when $k$ is finite, the method of Gabber can be used to prove independence of $l$ for weightless cohomology (see \cite{Fuji02} or \cite{Zha09} Prop. 2.7 and the remarks following it).

### 1.6. Heuristic

The properties of $EH^*(X)$ strongly suggest (when $k = \mathbb{C}$) the following topological picture, which we have found a useful heuristic: There is a nice topological space $\hat{X}$ with a proper mapping $p : \hat{X} \to X$ which is an isomorphism over the nonsingular locus of $X$ and such that $R\pi_*\mathcal{Q}_X = EC_X$. The space $\hat{X}$ should be constructed out of any resolution of singularities of $X$ but should be independent of the choice up to simple homotopy equivalence in a “stratumwise” manner. (This independence statement is not elementary. For example, if $X$ has isolated singularities one would expect the complement of the smooth locus in $\hat{X}$ to be the dual graph of components in a resolution of the singularity. This is, up to homotopy, independent of the choice of resolution (cf. \cite{Ste05} \cite{Thu07} \cite{ABW11} \cite{Pay13}), but this fact depends on the weak factorization theorem of birational geometry.) We call such a space $\hat{X}$ a weightless blowup of $X$. The properties of weightless cohomology should reflect properties of weightless blowups. We plan to investigate the existence of these...
spaces and study their properties in a sequel to this paper. This would, in particular, give an integral structure to \( EH^*(X) \), and we have other applications in mind for \( \bar{X} \). (Note that one nontrivial case is known: When \( X \) is the Baily-Borel compactification of a locally symmetric variety there is a natural \( \bar{X} \), namely the reductive Borel-Serre compactification.)

1.7. The outline of this paper is as follows: In Section 2 we discuss preliminaries on triangulated categories. Subsection 2.1 is devoted to general facts about \( t \)-structures, while 2.2 is devoted to Morel’s construction. In Section 3 we construct the required \( t \)-structures \( (wD\leq D, wD\geq D) \) and discuss some of their basic properties. We also introduce the complexes \( EC^D_X \) on any scheme \( X \) and study their properties. In Section 4 we construct pullbacks and establish the other main properties of \( EH^*(X) \). Section 5 is devoted to examples.

**Notation.** All schemes will be reduced, separated, of finite type over a fixed field \( k \), which will be either a finite field or \( \mathbb{C} \) (except possibly in Section 2.1). We will let \( D^b(X) \) denote the bounded derived category of mixed complexes \( D^b_{\text{mix}}(X, \mathbb{Q}_l) \) if \( k \) is finite with \( \text{char}(k) \neq l \) (as in [BBD82]) or the derived category of mixed Hodge modules \( D^b_{\text{MHM}}(X) \) (in the sense of Saito, see [Sai99] [Sai90], especially §4 of the latter) if \( k = \mathbb{C} \). In the latter case an assertion like “\( F \) is a local system” should be interpreted as the corresponding statement for its constructible realization \( \text{rat}(F) \) (see [Sai99, 1.3]).

Treated the cases \( k \) finite and \( k = \mathbb{C} \) simultaneously leads to one notational problem: If \( k = \mathbb{C} \) the notation \( ^pH^i \) is to be read as the cohomology functor for the “standard” \( t \)-structure on mixed Hodge modules, which is usually simply denoted \( H^i \). If \( k = \mathbb{C} \) the notation \( H^i \) is to be read as \( cH^i \), the cohomology functor corresponding to the analogue of the classical \( t \)-structure on \( D^b_{\text{MHM}}(X) \) [Sai90, 4.6].

Finally, by the constant sheaf \( \mathbb{Q}_X \), we will mean the constant \( l \)-adic sheaf \( \mathbb{Q}_l \) on \( X \) if \( k \) is finite, or the object \( \mathbb{Q}^H_X := a_X^*\mathbb{Q}^H \) where \( a_X : X \to \text{Spec} \mathbb{C} \) and \( \mathbb{Q}^H \) is the constant object, if \( k = \mathbb{C} \).

By an abuse of terminology, “triangle” will always mean “distinguished triangle”.

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2. Preliminaries

2.1. Triangulated categories and \( t \)-structures. In this section, we recall some basic properties of gluing of \( t \)-structures in the form we will apply them later. The standard reference for \( t \)-structures is [BBD82].
2.1.1. Notation. We use the following notation:

\[ t = (t^0, t^0) \quad \text{A } t\text{-structure ([BBDS2 Section 1]) on the triangulated category } D \]

\[ t[-r] = (t^r, t^r) \quad \text{The } t\text{-structure } (t^0[-r], t^0[-r]) \]

\[ t^- \quad \text{The } t\text{-structure } (\tau^0_{\leq 0}, \tau^0_{\geq 0}), \text{ where } +\infty D^0 = D \text{ and } +\infty D^0 = \{0\} \]

We will also replace a \( t\)-structure by the corresponding index \( (e.g. \text{ use } t^-_0 \text{ instead of } t^-_0) \) if it simplifies notation.

If the triangulated category is not specified, the \( t\)-structure should be thought of as on the category \( D^b(X) \) for an appropriate \( X \). If we want to stress the scheme \( X \) in such cases, we will use notation \( (t^0 X, t^0 X) \) for the \( t\)-structure.

We recall the following basic facts about \( t\)-structures:

2.1.2. Lemma. If \( t, t' \) are two \( t\)-structures on the same underlying triangulated category such that \( t^0 \subset t^0 \) (or equivalently \( t^0 \subset t^0 \)), then for all objects \( F \) the following natural maps are isomorphisms:

\[ t^-_0 \tau^-_0 F \quad \tau^-_0 \quad t^-_0 \tau^-_0 F \]

\[ t^-_0 t^-_0 \tau^-_0 \quad t^-_0 \tau^-_0 \tau^-_0 \]

Proof. See [BBDS2 1.3.4], where this lemma is stated in the case \( t = t'\lceil n \) for some \( n > 0 \). Even though we need this slightly more general case for use with Morel’s \( t\)-structures, the proof remains the same. \( \square \)

2.1.3. Notation. By a stratification of a scheme \( X \) we mean a collection of locally closed subschemes \( S_0, S_1, \ldots, S_r \) with \( S_i \) open in \( X - \bigcup_{0 \leq j < i} S_j \), such that closure of any stratum is a union of strata. We will also denote \( S_0 \) by \( U \). We will always assume each stratum to be irreducible. We will often denote the stratification \( (S_0, S_1, \ldots, S_r) \) by \( S \).

We also let \( U_k := \bigcup_{j \leq k} S_j \) be the open subschemes, and \( Z_k := X - U_k \) be their closed complements. We let \( f_k : U_k \to X \) and \( i_k : Z_k \to X \) denote the corresponding immersions. Let \( f_k : S_k \to X \) denote the locally closed immersion.

Further let \( f_{ik} : U_i \to U_k \) and \( f_{ik} : (U_k - U_i) \to U_k \) and \( f_{ik} : S_i \to U_k \) denote the natural maps for \( i \leq k \).

The following lemma summarizes results on gluing of \( t\)-structures from [BBDS2].

2.1.4. Lemma. We assume the situation of 2.1.3. Let there be \( t\)-structures \( t_i = (D^0(S_i), D^0(S_i)) \) on \( S_i \), and the corresponding truncations be \( (\tau^0_{\leq 0}, \tau^0_{\geq 0}) \). Define:

\[ D^0(X) = \{ F | f_i^* F \in D^0(S_i), \forall i \} \]

\[ D^0(X) = \{ F | f_i^* F \in D^0(S_i), \forall i \} \]

Then:

(1) \( (D^0(X), D^0(X)) \) is a \( t\)-structure. We denote it by \( t_0, t_1, \ldots, t_r \).

(2) The order of gluing is irrelevant (e.g. \( \{t_0, \ldots, t_r\} = \{t_0, t_1, t_2, \ldots, t_r\}, \text{ etc.} \) ).
2.1.5. Lemma. Let the notation be as in the last lemma. Then for all \( i \) we have the triangles:
\[
\begin{align*}
\sigma^i_{\leq 0} \mathcal{F} & \to \mathcal{F} \to Rf_i i^* \tau_{\leq 0} \mathcal{F} \\
Rf_i i^* \tau_{\leq 0} \mathcal{F} & \to \mathcal{F} \to \sigma^i_{> 0} \mathcal{F}
\end{align*}
\]

We also have the following isomorphisms:
\[
\begin{align*}
Rf_i i^* \tau_{\leq 0} \mathcal{K} & \cong \sigma^i_{\leq 0} Rf_i \mathcal{K} \\
Rf_i i^* \tau_{> 0} \mathcal{K} & \cong \sigma^i_{> 0} Rf_i \mathcal{K}
\end{align*}
\]

Proof. The first two triangles are proved as in \cite[3.3.4]{Mor08}. For the last two isomorphisms consider the triangle:
\[
Rf_i i^* \tau_{\leq 0} \mathcal{K} \to Rf_i \mathcal{K} \to Rf_i i^* \tau_{> 0} \mathcal{K}
\]

obtained by applying the triangulated functor \( Rf_i \) to the standard triangle for \( t_i \). Let \( t' := [\ldots, +\infty, t_i, +\infty, \ldots] \). Therefore \( \sigma^i_{\leq 0} = t' \tau_{\leq 0} \) by definition.

Then, since \( f_i^* Rf_i i^* \tau_{\leq 0} \mathcal{K} = i^* \tau_{\leq 0} \mathcal{K} \in t'_i D^{\leq 0} \), the first term is in \( t'_i D^{\leq 0} \). Also for the last term, \( f_j^* Rf_i i^* \tau_{> 0} \mathcal{K} = 0 \) for \( j \neq i \), while for \( j = i \), \( f_j^* Rf_i i^* \tau_{> 0} \mathcal{K} = i^* \tau_{> 0} \mathcal{K} \in i^! D^{> 0} \), and hence the last term is in \( t'_i D^{> 0} \). Therefore, we must have \( Rf_i i^* \tau_{\leq 0} \mathcal{K} \cong t'_i \tau_{\leq 0} Rf_i \mathcal{K} \) as required. The second isomorphism is just the dual statement. \( \square \)

2.2. Morel’s \( t \)-structures and weights. In \cite[Section 3]{Mor08}, S. Morel defines certain \( t \)-structures on the category of mixed \( l \)-adic complexes \( D^b_{\text{MHM}}(X, \mathbb{Q}_l) \) on a scheme \( X \) over a finite field \( k \) using the notion of weights. We recall her key definitions below, and develop some properties of these \( t \)-structures which we will need in the sequel. Morel’s construction is quite general and immediately carries over to any reasonable category of mixed sheaves (\cite{Sa00}), for example the derived category of mixed Hodge modules of Saito. Thus we will use the analogues of Morel’s results for \( k = \mathbb{C} \) as well.

2.2.1. Notation. As is standard, we let \((p D^{\leq 0}, p D^{> 0})\) denote the perverse \( t \)-structure for middle perversity (\cite[Section 2]{BBDS2}), and we let \( p H^i \) denote the cohomology functors with respect to this perverse \( t \)-structure. We also let \((D^b_{\leq a}, D^b_{> a})\) denote bounded complexes of weight \( \leq a \) (resp. \( > a \)) in the sense of \cite[Section 5]{BBDS2}. We also let \((D^{\leq 0}, D^{> 0})\) denote the standard \( t \)-structure whose core is the abelian category of sheaves. \( \uparrow \)

For a fixed scheme \( X \) and any \( a \in \mathbb{Z} \), we define:
\[
\begin{align*}
^w D^{\leq a}(X) & := \text{full subcategory of } D^b(X) \text{ with objects } \mathcal{F} \text{ s.t. } \forall i, p^i H^i(\mathcal{F}) \in D^b_{\leq a} \\
^w D^{\geq a}(X) & := \text{full subcategory of } D^b(X) \text{ with objects } \mathcal{F} \text{ s.t. } \forall i, p^i H^i(\mathcal{F}) \in D^b_{\geq a}
\end{align*}
\]

\( \uparrow \)We remind the reader that in the case \( k = \mathbb{C} \) with \( D^b(X) = D^b \text{MHM}(X) \), \((p D^{\leq 0}, p D^{> 0})\) is the usual \( t \)-structure, \( p H^i \) is the usual cohomology functor and \((D^{\leq 0}, D^{> 0})\) is the analogue of the classical \( t \)-structure, cf. \cite[4.6]{Sa00}.
The central result of Morel’s construction is:

2.2.2. **Proposition.** For any $X$, and any integer $a$, we have:

1. $wD^{\leq a}$ and $wD^{\geq a}$ are subcategories stable under shifts and extensions inside $D^b(X)$. In particular they are triangulated subcategories of $D^b(X)$.
2. If (1) is the Tate twist, $wD^{\leq a}(1) = wD^{\leq a-2}$ and $wD^{\geq a}(1) = wD^{\geq a-2}$.
3. If $\mathcal{F} \in wD^{\leq a}$ and $\mathcal{G} \in wD^{\geq a+1}$, then $R\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$.
4. $wD^{\leq a}$ and $wD^{\geq a+1}$ form a $t$-structure on $D^b(X)$.

**Proof.** See [Mor08, Proposition 3.1.1]. \( \square \)

2.2.3. **Definition.** A complex of sheaves $\mathcal{F} \in D^b(X)$ is *lisse* on $X$ if the cohomology sheaves $H^i(X)$ are local systems for each $i$.

2.2.4. **Lemma.** Let $f : T \to S$ be a morphism of schemes:

1. If $f$ has fibres of dimension less than or equal to $d$, then, we have:
   
   \[
   Rf_!(wD^{\leq a}(T)) \subset wD^{\leq a+d}(S) \\
   f^*(wD^{\leq a}(S)) \subset wD^{\leq a+d}(T) \\
   Rf_!(wD^{>a}(T)) \subset wD^{>a-d}(S) \\
   f^!(wD^{>a}(S)) \subset wD^{>a-d}(T)
   \]

2. Assume $S, T$ are smooth irreducible of dimension $d_S, d_T$. If a complex $\mathcal{L}$ is lisse over $S$, then

   \[
   \mathcal{L} \in wD^{\leq a}(S) \Rightarrow f^*\mathcal{L} \in wD^{\leq a-d_S+d_T}(T) \\
   \mathcal{L} \in wD^{>a}(S) \Rightarrow f^!\mathcal{L} \in wD^{>a-d_S+d_T}(T) \\
   \mathcal{L} \in wD^{\leq a}(S) \Rightarrow f^*\mathcal{L} \in wD^{\leq a-d_T+d_S}(T) \\
   \mathcal{L} \in wD^{>a}(S) \Rightarrow f^!\mathcal{L} \in wD^{>a-d_T+d_S}(T)
   \]

**Proof.** (1) is contained in [Mor08 3.1.3(v)].

(2) It is enough to argue for local systems (since $wD^{\leq 0}$, etc. are triangulated categories, $pH^i d_S \mathcal{L} = (H^i \mathcal{L})[d_S]$ and we can use induction on number of nonzero $H^i$). Now for the first implication:

\[
\mathcal{L} \in wD^{\leq a}(S) \text{ and is a local system} \\
\Rightarrow \mathcal{L}[d_S] \in D_{\leq a}[d_S] \text{ (as $\mathcal{L}[d_S]$ is perverse, containment by definition)} \\
\Rightarrow \mathcal{L} \in D_{\leq a-d_S}(S) \\
\Rightarrow f^*\mathcal{L} \in D_{\leq a-d_T}(T) \text{ (by [BBDS82 5.1.14])} \\
\Rightarrow f^*\mathcal{L}[d_T] \in D_{\leq a-d_S+d_T}(T) \\
\Rightarrow f^*\mathcal{L}[d_T] \in wD^{\leq a-d_S+d_T}(T) \text{ (as $f^*\mathcal{L}[d_T]$ is perverse)} \\
\Rightarrow f^*\mathcal{L} \in wD^{\leq a-d_S+d_T}(T)
\]
as required. The second implication follows from first using duality. The third implication follows in the same way as the first noting that in our situation, if $\mathcal{L}$ is
pure of weight $a$, so is $f^*\mathcal{L}$ ([Del80 1.2.5]). Therefore if $\mathcal{L}$ has weights $> a$, so has $f^*\mathcal{L}$. The final implication is dual to the third.

2.2.5. **Notation.** Fix a stratification $S = (S_0, \ldots, S_n)$ of a scheme $X$, and let $\bar{d}^S = (d_0, \ldots, d_n)$ (or simply $\bar{d}$) stand for an $n+1$-tuple of integers indexed by the strata of $S$ (or simply $\bar{d}$). Therefore, we get a $t$-structure on each stratum $(wD^{\leq d_i}, wD^{> d_i+1})$. We can in fact allow $\bar{d}$ to take values in $\mathbb{Z} \cup \{+\infty, -\infty\}$, by using notation of 2.1.1 for $t$-structures $\pm \infty$.

These can then be glued using 2.1.4 to obtain a $t$-structure $(wD^{\leq \bar{d}}, wD^{> \bar{d}})$. We let $(w^{\leq \bar{d}}, w^{> \bar{d}})$ stand for truncations with respect to this $t$-structure.

2.2.6. **Lemma.** If $\bar{d} = (d, d, \ldots, d)$, then $wD^{\leq \bar{d}} = wD^{d}$ and $wD^{> \bar{d}} = wD^{d}$.

**Proof.** See [Mor08 3.3.3].

2.2.7. **Lemma.** Let $S, T$ be stratifications of schemes $X, Y$. Also let $\bar{d}^S, \bar{d}^T$ be a tuple of integers indexed by the stratifications $T$ and $S$ respectively. Then, $\mathcal{F} \in wD^{\leq \bar{d}^S}, \mathcal{G} \in wD^{\leq \bar{d}^T} \Rightarrow \mathcal{F} \boxtimes \mathcal{G} \in wD^{\leq \bar{d}^S}w^{\bar{d}^T}$
$\mathcal{F} \in wD^{> \bar{d}^S}, \mathcal{G} \in wD^{> \bar{d}^T} \Rightarrow \mathcal{F} \boxtimes \mathcal{G} \in wD^{> \bar{d}^S}w^{\bar{d}^T}$
where by $\bar{d}^S \boxtimes \bar{d}^T$ we mean the tuple subordinate to stratification $\{S_i \times T_j | i, j\}$ on $X \times Y$ which takes value $d_i^S + d_j^T$ on the stratum $S_i \times T_j$.

**Proof.** It is enough to show that $f^*_{ij}(\mathcal{F} \boxtimes \mathcal{G}) = f^*_{ij}\mathcal{F} \boxtimes f^*_{ij}\mathcal{G} \in wD^{\leq d_i^S + d_j^T}$. We let $\mathcal{F}_i := f^*_{ij}\mathcal{F}$, $\mathcal{G}_j := f^*_{ij}\mathcal{G}$. Now since our complexes are bounded, there is a triangle, for some large $n$:

$$pH^n\mathcal{F}_i \boxtimes \mathcal{G}_j \rightarrow \mathcal{F}_i \boxtimes \mathcal{G}_j \rightarrow p_{\tau > -n}\mathcal{F}_i \boxtimes \mathcal{G}_j \rightarrow$$

Now by induction on the number of nonzero perverse cohomology objects and using that $wD^{\leq a}$ is closed under taking triangles and perverse truncations for any integer $a$, it is enough to assume that $\mathcal{F}_i, \mathcal{G}_j$ are perverse, and hence, by definition of Morel’s $t$-structure, in $D^{\leq d_i^S}$ (resp. $D^{> d_i^T}$). But then $\mathcal{F}_i \boxtimes \mathcal{G}_j$ is perverse as well ([BBD82 4.2.8]), and since $\mathcal{F}_i \boxtimes \mathcal{G}_j \in D^{\leq d_i^S + d_j^T}$ ([BBD82 5.1.14]), we have $\mathcal{F}_i \boxtimes \mathcal{G}_j \in wD^{\leq d_i^S + d_j^T}$ as required. The other statement follows by duality.

2.2.8. **Lemma.** Let $S$ be a stratification on a scheme $X$. Let $\bar{d}, \bar{d}'$ be tuple of integers indexed by the stratification $S$. Assume in addition that for all strata $f_i : S_i \rightarrow X$ in $S$, $S_i$ is smooth and $f_i^*\mathcal{F}, f_i^*\mathcal{F}$, as well as $f_i^*\mathcal{G}$ and $f_i^*\mathcal{G}$, are isisse. Then $\mathcal{F} \in wD^{\leq \bar{d}}, \mathcal{G} \in wD^{\leq \bar{d}'} \Rightarrow \mathcal{F} \otimes \mathcal{G} \in wD^{\leq \bar{d} \otimes \bar{d}'}$
where by $\bar{d} \otimes \bar{d}'$ we mean the tuple which takes value $d_i + d_i' - \dim(S_i)$ on the stratum $S_i$.

**Proof.**It is enough to show that $f^*_{ij}(\mathcal{F} \otimes \mathcal{G}) = f^*_{ij}\mathcal{F} \otimes f^*_{ij}\mathcal{G} \in wD^{\leq d_i + d_i' - \dim S_i}$. We let $\mathcal{F}_i := f^*_{ij}\mathcal{F}$, $\mathcal{G}_i := f^*_{ij}\mathcal{G}$. Now since our complexes are bounded, there is a triangle, for some large $n$:

$$\mathcal{H}^{-n}\mathcal{F}_i \otimes \mathcal{G}_i \rightarrow \mathcal{F}_i \otimes \mathcal{G}_i \rightarrow \tau_{> -n}\mathcal{F}_i \otimes \mathcal{G}_i \rightarrow$$

Now by induction on the number of nonzero perverse cohomology objects and using that $wD^{\leq a}$ is closed under taking triangles, perverse truncations and shifts, and that for the
complex $\mathcal{F}_i$, $\mathcal{H}^{-n}\mathcal{F}_i = (\mathcal{p}H^n\mathcal{F}_i)[-n]$ (since all the cohomologies are given to be local systems on the smooth variety $S_i$) it is enough to assume that $\mathcal{F}_i, \mathcal{G}_i$ are local systems, and hence, by definition of Morel’s $t$-structure, in $D^b_{\leq d_i - \dim S_i}$ (resp. $D^b_{\leq d'_i - \dim S_i}$). But then $\mathcal{F}_i \otimes \mathcal{G}_i$ is a local system as well, and since $\mathcal{F}_i \otimes \mathcal{G}_i \in D^b_{\leq d_i + d'_i - 2 \dim S_i}$ ([BBD82, 5.1.14]), we have $\mathcal{F}_i \otimes \mathcal{G}_i \in wD_{\leq d_i + d'_i - \dim S_i}$ as required.

2.2.9. Lemma. For a stratification $S = (S_0, ..., S_i, ..., S_n)$ of a scheme $X$ the functor $w_{\leq (+\infty, ..., +\infty, d, +\infty, ..., +\infty)}$ is left exact with respect to the standard $t$-structure, i.e. it takes $D^{\geq 0}$ to $D^{\geq 0}$.

Proof. We will use Noetherian induction on the underlying scheme $X$.

If $\dim X = 0$ (base case of induction), then $X$ is a union of points. The truncation will be left exact, if it is so on each point, so we can even assume that $X$ is a point. Then the only possible stratification is $(X)$ and truncation is $w_{\leq d}$ with $d \in \mathbb{Z} \cup \{\infty\}$. The claim is now immediate.

Reduction: We first reduce the problem to showing that $w_{\leq d}$ is left exact on $S_i$.

Let $f : S_i \hookrightarrow X$ be a stratum. Then for any complex $\mathcal{F} \in D^{\geq 0}$ we have the triangle:

$$w_{\leq (+\infty, ..., +\infty, d, +\infty, ..., +\infty)} \mathcal{F} \to \mathcal{F} \to Rf_* w_{\geq d} f^* \mathcal{F} \to$$

and hence a long exact sequence:

$$\to H^{i-1}(Rf_* w_{\geq d} f^* \mathcal{F}) \to H^i(w_{\leq (+\infty, ..., +\infty, d, +\infty, ..., +\infty)} \mathcal{F}) \to H^i(\mathcal{F}) \to H^i(Rf_* w_{\geq d} f^* \mathcal{F}) \to$$

Hence if $H^{i-1}(Rf_* w_{\geq d} f^* \mathcal{F}) = 0$ for $i < 0$, we have an injection

$$0 \to H^i(w_{\leq (+\infty, ..., +\infty, d, +\infty, ..., +\infty)} \mathcal{F}) \to H^i(\mathcal{F}) \cong 0 \quad \text{(for } i < 0) \text{.}$$

Therefore, it is enough to show that $H^i(Rf_* w_{\geq d} f^* \mathcal{F}) = 0$ for $j = i - 1 < -1$, that is, it is enough to show that $Rf_* w_{\geq d} f^* \mathcal{F} \in D^{\geq -1}$. Since $f^*$ is exact while $Rf_*$ is left exact, it is enough to show that $w_{\geq d}(D^{\geq 0}) \subset D^{\geq -1}$ on $S_i$. Now, for any complex $\mathcal{F}$ on $S_i$, using the triangle:

$$w_{\leq d} \mathcal{F} \to \mathcal{F} \to w_{\geq d} \mathcal{F} \to$$

implies a long exact sequence

$$\to H^i(\mathcal{F}) \to H^i(w_{\geq d} \mathcal{F}) \to H^{i+1}(w_{\leq d} \mathcal{F}) \to H^{i+1}(\mathcal{F}) \to$$

and hence if $\mathcal{F} \in D^{\geq 0}$, we have that $H^i(w_{\geq d} \mathcal{F}) \cong H^{i+1}(w_{\leq d} \mathcal{F})$ for $i < -1$. Therefore it is enough to show that $H^j(w_{\leq d} \mathcal{F}) = 0$ for $j = i + 1 < 0$ whenever $\mathcal{F} \in D^{\geq 0}$, in other words, that $w_{\leq d}$ is left exact.

Proving the reduced statement: Now assume $X$ is arbitrary. If $S_i$ is a proper subset of $X$ this follows from Noetherian induction (using the stratification $(S_i)$ on $S_i$). If $S_i = X$, we restratify $S_i = U \sqcup Z$, with the property that $\mathcal{F}|_U$ is lisse and $U$ is smooth.

Now $w_{\leq d} = w_{\leq (d,d)} = w_{\leq (\infty,d)} w_{\leq (d,\infty)}$ on $S_i$ by 2.2.6 and $w_{\leq (\infty,d)}$ is left exact by what we have done so far. Therefore, it is enough to show that $w_{\leq (d,\infty)} \mathcal{F} \in D^{\geq 0}$.

If $j : U \to S_i$ denotes the open immersion, as before we have the triangle:

$$w_{\leq (d,\infty)} \mathcal{F} \to \mathcal{F} \to Rj_* w_{\geq d} j^* \mathcal{F} \to$$

But now $j^* \mathcal{F}$ is lisse, and hence we have $H^i(\mathcal{F}) = \mathcal{p}H^{i+\dim S_i}(\mathcal{F})[-\dim S_i]$. Over the smooth set $U$, $w_{\geq d}$ is perverse exact and takes local systems to local systems, therefore
Proof. \( w_{d} F \in D^{\geq 0} \), and consequently \( R_{j_{*}} w_{d} F \in D^{\geq 0} \). But then using the long exact sequence of cohomology:

\[ \rightarrow H^{i-1}(R_{j_{*}} w_{d} F) \rightarrow H^{i}(w_{d, \infty} F) \rightarrow H^{i}(F) \rightarrow H^{i}(R_{j_{*}} w_{d} F) \rightarrow \]

we see that \( H^{i}(w_{d, \infty} F) \cong H^{i}(F) \cong 0 \) for \( i < 0 \), that is \( w_{d, \infty} F \in D^{\geq 0} \) as required. \( \square \)

2.2.10. Lemma. Let us fix a stratification \( S = (U, Z) \) of \( X \). Then the functor \( w_{\leq (\infty, d)} F \) is perverse right exact, i.e. \( w_{\leq (\infty, d)} F \in p D^{\leq 0} \subset p D^{\leq 0}. \)

Proof. Fix \( F \in p D^{\leq 0} \) and let \( E = w_{\leq (\infty, d)} F \). Consider the triangle of 2.1.5:

\[ E \rightarrow F \rightarrow i_{*} w_{d} F \rightarrow \]

Applying \( j^{*} \) we see that \( j^{*} F = j^{*} F \in p D^{\leq 0} \). Applying \( i^{*} \) we see that \( i^{*} E \cong w_{d} i^{*} F \). But \( i^{*} F \in p D^{\leq 0} \), while \( w_{d} \) is perverse exact. Hence \( i^{*} E \in p D^{\leq 0} \) as well. Hence by definition of the perverse t-structure as a glued t-structure, we see that \( E \in p D^{\leq 0}. \) \( \square \)

2.2.11. Lemma. Let \( S = (U, Z) \) be a stratification of \( X \). Then for any \( a \in \mathbb{Z} \), the functor \( w_{d, b} F \) preserves the subcategory of complexes of weight \( \leq a \).

Proof. If \( F \in D^{b}(X) \) has weights \( \leq a \), then \( i^{*} w_{d, b} F = w_{d} i^{*} F \) we see that \( p H^{k}(i^{*} w_{d, b} F) = w_{d} p H^{k}(i^{*} F) \) has weights \( \leq \min(b, k + a) \leq k + a \) for all \( k \) (since \( i^{*} F \) has weights \( \leq a \)). So \( i^{*} w_{d, b} F \) has weights \( \leq a \). Since \( j^{*} w_{d, b} F = j^{*} F \) also has weights \( \leq a \), the pointwise criterion for weights (cf. [BBD82 5.1.9] or [Sai90 4.6.1]) implies that \( w_{d, b} F \) has weights \( \leq a \).

\( \square \)

2.2.12. Lemma. Let \( X \) be a scheme and let \( F \) denote an irreducible perverse sheaf, pure of weight \( d \), whose support is the whole of \( X \). Then on any proper closed subscheme \( i : Z \hookrightarrow X \), \( i^{*} F \in w D^{\leq d-1}. \)

Proof. We have \( i^{*} F \in p D^{\leq 0} \), hence for \( i > 0 \), \( p H^{i}(i^{*} F) = 0 \). Also \( i^{*} F \in D_{d}^{b}. \) Hence, \( p H^{i}(i^{*} F) \in D_{d+i}^{b} \), and for \( i < 0 \), \( D_{d+i}^{b} \subset D_{d}^{b} \). Hence for \( i \neq 0 \), \( p H^{i}(i^{*} F) \in D_{d}^{b} \), and it only remains to verify the same for \( p H^{0}(i^{*} F) \). But for a perverse sheaf, just as in the previous lemma, we have a surjection:

\[ F = p H^{0} F \rightarrow p H^{0}(i^{*} F) = i^{*} p H^{0}(i^{*} F) \]

However since \( F \) is irreducible and has support on whole of \( X \) and \( Z \) is proper, this implies that \( i^{*} p H^{0}(i^{*} F) = 0 \). Hence also \( p H^{0}(i^{*} F) = 0 \) as required. Therefore by definition of Morel’s t-structure, \( i^{*} F \in w D^{\leq d-1} \) and we are done. \( \square \)

2.2.13. Lemma. Let \( X \) be a scheme with \( j : U \hookrightarrow X \) open. Let \( F \in w D^{\leq a}(U) \cap w D^{\geq a}(U) \) and let \( S = (S_{0}, ..., S_{r}) \) be a stratification with \( S_{0} = U \). Then for any \( r + 1 \)-tuple \( d = (a, d_{1}, ..., d_{r}) \) we have a natural isomorphism

\[ w_{(a, d_{1}, ..., d_{r})} R j_{*} F \cong w_{(a, d_{1}+1, ..., d_{r}+1)} j_{!} F. \]

Proof. This is [Mor08 Prop. 3.4.2]. \( \square \)

3. Weight-truncated complexes

3.1. Definition of the weight-truncated complexes \( EC_{X}^{D} \).
3.1.1. Notation. We let \( S := \{ S | S = (S_0, \ldots, S_r) \} \) be a stratification of \( X \) in smooth locally closed equidimensional subschemes be partially ordered by \( \leq \), where

\[ S \leq S' \text{ if } S' \text{ is a refinement of } S \]

We also use the notations of \ref{2.1.3} with superscript to indicate the relevant stratification (e.g. \( S w_{\leq d} \)). Superscripts will also be used to specify the scheme under consideration. All our stratifications will be in smooth equidimensional subschemes unless specified otherwise.

Further, we let \( D : \{ 0, \ldots, \dim X \} \rightarrow \mathbb{Z} \) denote any function, we will impose additional restrictions later. Let \( Sw_{\leq D} \) (respectively \( Sw_{> D} \)) denote the functor \( w_{\leq (D(d_0), \ldots, D(d_r))} \) (respectively \( w_{> (D(d_0), \ldots, D(d_r))} \)) where \( d_i \) is the dimension of \( S_i \). By abuse of notation we will also use \( D(Y) \) for \( D(\dim Y) \).

3.1.2. Notation. For any \( d \in \mathbb{Z} \) we say a function \( D : \{ 0, \ldots, d \} \rightarrow \mathbb{Z} \) is a monotone step function if \( D \) is monotone, and \( |D(j) - D(j - 1)| \leq 1 \) for all \( 1 < j \leq d \). In particular, we say \( D \) is non-increasing (resp. non-decreasing) step function if it is non-increasing (resp. non-decreasing).

3.1.3. Theorem. For any scheme \( X \), if \( D : \{ 0, \ldots, \dim X \} \rightarrow \mathbb{Z} \) is a non-decreasing step function, then for any \( G \in D^b(X) \),

\[ w_{\leq D} G := \lim_{s \in S} S w_{\leq D} G \]

is well-defined. In fact \( w_{\leq D} G = S w_{\leq D} G \) for all \( S \) sufficiently fine. Similarly, if \( D \) is a non-increasing step function, then:

\[ w_{> D} G := \lim_{s \in S} S w_{> D} G \]

and \( w_{> D} G = S w_{> D} G \) for all \( S \) sufficiently fine as before.

Proof. Consider \( D \) non-decreasing. Let \( S \) be a stratification in smooth subschemes such that \( f_i^* G \) and \( f_i^*(S w_{\leq D} G) \) are lisse on stratum \( S_i \) where \( f_i : S_i \hookrightarrow X \) is the locally closed inclusion (we will show below that such a stratification exists).

Then it is enough to show that if \( T \geq S \) is another stratification in smooth subschemes, then \( S w_{\leq D} G = T w_{\leq D} G \), under the natural maps as described below. It follows from \ref{2.2.6}

\[ S w_{\leq D} = w_{\leq (D(d_0^S), D(d_0^S), \ldots; D(d_1^S), D(d_1^S), \ldots)} \]

where truncation on the right is with respect to \( T \) and we use \( D(d_j^S) \) on \( T_i \), if \( T_i \subset S_j \).

Now since, in the above situation, \( \dim(T_i) \leq d_j^S \), hence \( D(T_i) \leq D(d_j^S) \) and hence we have \( T w_{\leq D} = T w_{\leq D} S w_{\leq D} \) \( \ref{2.1.2} \), and this gives the morphisms with respect to which the limit is taken. It is enough to show that

\[ S w_{\leq D} G \in w D^{\leq (D(d_0^S), D(d_1^S), \ldots)} \]

To show this, we only need to show that, \( f_i^{T*}(S w_{\leq D} G) \in w D^{\leq D(d_i^T)} \). Now \( f_i^{T*} = f_{i_j}^{T*S} f_{j_i}^{S*} \), where \( f_{i_j}^{T*S} \) denotes the inclusion \( T_i \rightarrow S_j \). Since by definition of \( S w_{\leq D} \), \( f_{i_j}^{T*S}(S w_{\leq D} G) \in w D^{\leq D(d_i^T)} \), we have by \ref{2.2.4} \( f_i^{T*}(S w_{\leq D} G) \in w D^{\leq D(d_i^T)} \). This gives the morphisms with respect to which the limit is taken. It is enough to show that

\[ S w_{\leq D} G \in w D^{\leq D(d_i^T)} \]

Now to prove the existence of such a stratification \( S \) we proceed by Noetherian induction. Since over a point all complexes are lisse, the base case is immediate. More generally, given
a complex $\mathcal{G}$, let $j : U \hookrightarrow X$ denote an open subset such that $j^*(\mathcal{G})$ is lisse, which exists since $\mathcal{G}$ is constructible. Then we have a triangle:

$$w_{\leq (D(\dim X),\infty)} \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow j_* w_{> D(\dim X)} j^* \mathcal{G} \longrightarrow$$

and we let $\mathcal{H} := w_{\leq (D(\dim X),\infty)} \mathcal{G}$. By the induction hypothesis we have a stratification $S'$ of $\mathcal{Z} := X - U$ such that $f_i^* (\mathcal{H}|_Z)$ as well as $f_i^* (S' w_{\leq D}(\mathcal{H}|_Z))$ are lisse for any stratum in $S'$. Furthermore, by what we have seen above, if we replace $S'$ by a finer stratification, $\mathcal{H}|_Z$ and $S' w_{\leq D}(\mathcal{H}|_Z)$ don’t change. Therefore we may further assume that $f_i^* G|_Z$ is also lisse for all strata in $S'$.

Now we let $S$ be the stratification of $X$ given by $U$ and the strata in $S'$. We have a triangle:

$$S w_{\leq D} \mathcal{G} \equiv S w_{(\infty,D)} \mathcal{H} \longrightarrow \mathcal{H} \longrightarrow i_* S' w_{> D}(\mathcal{H}|_Z) \longrightarrow$$

where $i : \mathcal{Z} \hookrightarrow X$. If $f_i' : S_i \hookrightarrow Z$ is a stratum in $S'$, pullback by $f_i := i \circ f_i'$ gives a triangle:

$$f_i^* (S w_{\leq D} \mathcal{G}) \longrightarrow f_i^* (\mathcal{H}|_Z) \longrightarrow f_i^* S' w_{> D}(\mathcal{H}|_Z) \longrightarrow$$

Comparing this with the triangle obtained by applying $f_i^*$ to the triangle $S' w_{\leq D}(\mathcal{H}|_Z) \rightarrow \mathcal{H}|_Z \rightarrow S' w_{> D}(\mathcal{H}|_Z)$, we get that $f_i^* (S w_{\leq D} \mathcal{G}) \equiv f_i^* (S w_{\leq D}(\mathcal{H}|_Z))$, which is lisse by our assumptions on $S'$. For the open stratum $j : U \hookrightarrow X$ we have the triangle:

$$j^* (S w_{\leq D} \mathcal{G}) \longrightarrow j^* \mathcal{G} \longrightarrow w_{> D(\dim X)} j^* \mathcal{G} \longrightarrow$$

and hence $j^* (S w_{\leq D} \mathcal{G}) \equiv w_{\leq D(\dim X)} j^* \mathcal{G}$ is lisse, since $j^* \mathcal{G}$ is. In particular $S$ has the requisite properties.

The case of $S w_{> D}$ for non-increasing $D$ is similar, working with a stratification $S$ such that both $f_i^* (\mathcal{G})$ and $f_i^* (S w_{> D} \mathcal{G})$ are lisse for all strata $f_i : S_i \hookrightarrow X$ in $S$. \hfill \Box

3.1.4. Definition. Fix a complex $\mathcal{G}$. For a non-decreasing step function we have from the proof above that $w_{\leq D} \mathcal{G} = S w_{\leq D} \mathcal{G}$ for some stratification $S$. We define $w_{> D} \mathcal{G} := S w_{> D} \mathcal{G}$. If $T$ is a refinement of $S$, using the morphism of triangles

$$T w_{\leq D} \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow T w_{> D} \mathcal{G} \longrightarrow$$

we see that the last vertical map exists and must be an isomorphism as well. Hence the definition does not depend on choice of the stratification (a priori up to non-canonical isomorphism, but see 3.1.7). Similarly we can work with non-increasing step functions $D$ as well, and define $w_{\leq D}$ for such functions.

3.1.5. Notation. We will let $\text{Id} := m \mapsto m$. Note that for any integer $a$ Morel’s truncation $w_{\leq a}$ coincides with $w_{\leq a}$ with $a$ being the constant function using 2.2.6. Hence we can also let an integer $a$ stand for the constant function $\bar{a}$ and this causes no confusion.

3.1.6. Definition. We say a stratification of a scheme $X$ is adapted to a complex $\mathcal{G} \in D^b(\mathcal{X})$ and a monotone step function $D$ if $f_i^* \mathcal{G}$, $f_i^* w_{\leq D} \mathcal{G}$, $f_i^* w_{> D} \mathcal{G}$ are all lisse on all strata $S_i$ for all $i$, where $f_i : S_i \hookrightarrow X$ denotes the locally closed immersion of a stratum.

By the proof of 3.1.3 we see that for any stratification $S$ adapted to $\mathcal{G}$ and any monotone step function $D$, we have the identity $S w_{\leq D} \mathcal{G} = w_{\leq D} \mathcal{G}$ and $S w_{> D} \mathcal{G} = w_{> D} \mathcal{G}$.

Clearly, given any finite collection of bounded complexes and a finite set of monotone step functions $D$, there is a stratification adapted to all of them (taking common refinements).
In the sequel, we will be often restrict to dealing with a finite set of monotone step functions $D$ and hence we will abuse notation and say that the stratification is adapted to $\mathcal{G}$ if it is adapted to $\mathcal{G}$ for all monotone step functions under consideration.

We also say a stratification is adapted to a scheme $X$ if it is adapted to $IC_X$ in the above sense (where $IC_X$ denotes the intersection complex on $X$).

**3.1.7. Proposition.** If $D$ is a monotone step function then $w_{\leq D}, w_{> D}$ are truncations with respect to a t-structure which we denote by $(w_{\leq D}, w_{> D})$.

**Proof.** We let $w_{\leq D} := \{G|G \equiv w_{\leq D}F, F \in D^b\}$ and $w_{> D} := \{G|G \equiv w_{> D}F, F \in D^b\}$ (i.e. full subcategories with these objects). By working at a stratification $S$ adapted to both $\mathcal{F}$ and $\mathcal{G}$ it easily follows that $Hom(w_{\leq D}F, w_{> D}G) = 0$ (since both truncations are with respect to an actual glued t-structure relative to $S$). The other axioms for a t-structure are almost immediate by definition. \hfill $\Box$

Let us record a few propositions which will be useful later:

**3.1.8. Proposition.** Let $\pi : Y \to X$ be a finite morphism. Let $D$ be any monotone step function. Then:

\[ \mathcal{G} \in w_{\leq D}(Y) \implies R\pi_*\mathcal{G} \in w_{\leq D}(X) \]

\[ \mathcal{G} \in w_{> D}(Y) \implies R\pi_*\mathcal{G} \in w_{> D}(X) \]

**Proof.** Fix a $\mathcal{G} \in w_{\leq D}(Y)$. By 3.1.12 we can find a smooth stratification $S^Y$ of $Y$ adapted to $\mathcal{G}$ and $S^X$ of $X$ adapted to $R\pi_*\mathcal{G}$ such that each stratum of $Y$ surjects onto some stratum of $X$. This image must be of the same dimension because $\pi$ is finite.

But then by dimension reasons, for each $i$, $\pi^{-1}S^X_i = \sqcup S^Y_j$ the right hand side being a disjoint finite union of some set of strata depending on $i$.

Therefore, if $f_i : S^X_i \hookrightarrow X$ and $g_j : S^Y_j \hookrightarrow Y$ are the inclusions,

\[ f_i^* R\pi_* \mathcal{G} \equiv \oplus_j R\pi_{ji*} g_j^* \mathcal{G} \]

where $\pi_{ji} : S^Y_j \to S^X_i$ is the restriction of $\pi$ to $S^Y_j$, and hence also finite. Since $\mathcal{G} \in w_{\leq D}(Y)$, $g_j^* \mathcal{G} \in w_{\leq D}(\dim S^X_i)$. Hence using 2.2.4 we see that $f_i^* R\pi_* \mathcal{G} \in w_{\leq D}(\dim S^X_i)$.

Since $\dim S^X_i = \dim S^Y_j$, we conclude that $R\pi_* \mathcal{G} \in w_{\leq D}(X)$.

The other case follows by duality since $\pi$ is proper. \hfill $\Box$

**3.1.9. Proposition.** Let $\pi : Y \to X$ be a smooth morphism. Then:

\[ \mathcal{G} \in w_{\leq \Id}(X) \implies \pi^* \mathcal{G} \in w_{\leq \Id}(Y) \]

\[ \mathcal{G} \in w_{> \Id}(X) \implies \pi^* \mathcal{G} \in w_{> \Id}(Y) \]

**Proof.** Fix $\mathcal{G} \in w_{\leq \Id}(X)$ and consider a stratification $S$ of $X$ adapted to $\mathcal{G}$. Consider the stratification of $Y$ with each stratum $T_j$ being some irreducible component of $\pi^{-1}S_i$ for some $i$.

Since $\mathcal{G}|_{S_i}$ is lisse on $S_i$, so is $\pi^* \mathcal{F}|_{T_j}$ on $T_j$. Since $S_i$ is smooth so are $T_j$. Hence, in particular, this stratification is adapted to $\pi^* \mathcal{F}$.

Therefore for $\pi|_{T_j} : T_j \to S_i$, 2.2.4 implies that $\pi^* \mathcal{G}|_{T_j} \in w_{\leq \dim T_j}$ if $\mathcal{G}|_{S_i} \in w_{\leq \dim S_i}$, and hence the first assertion. The proof for $w_{> \Id}$ is the same. \hfill $\Box$

The next proposition is the key input to construction of pullbacks:
3.1.10. Proposition. Let \( \pi : Y \to X \) be an arbitrary morphism. Then:

\[
\mathcal{G} \in \mathcal{w} D^{>\text{Id}}(Y) \implies R\pi_*\mathcal{G} \in \mathcal{w} D^{>\text{Id}}(X)
\]

\[
\mathcal{G} \in \mathcal{w} D^{\leq\text{Id}}(X) \implies \pi^*\mathcal{G} \in \mathcal{w} D^{\leq\text{Id}}(Y)
\]

Proof. The second statement follows using adjunction from the first, since \((\mathcal{w} D^{\leq\text{Id}}, \mathcal{w} D^{>\text{Id}})\) forms a t-structure. We now prove the first statement.

Suppose first that we have stratifications of \( S^Y, S^X \) adapted to \( \mathcal{G}, R\pi_*\mathcal{G} \) such that:

- Each stratum of \( S^Y \) surjects onto a stratum of \( S^X \).
- Restricting \( \pi \) to a single stratum of \( S^Y \), each fibre is of same dimension.

Then, we write \( S^Y_{ij} \to S^X_i \), for some finitely many \( j \) for each \( i \) (i.e. \( S^X_i \) is a stratum of \( X \) and \( S^Y_{ij} \) as \( j \) varies are the finitely many strata of \( \pi^{-1} S^X_i \)). Hence \( \mathcal{F} \in \mathcal{w} D^{>d^Y_{ij}}(S^Y_{ij}) \implies R\pi_*\mathcal{F} \in \mathcal{w} D^{>d^X_i}(S^X_i) \) (using \( \text{2.2.4} \) since fibres must be of dimension \( d^Y_{ij} - d^X_i \)). Using it for \( \mathcal{F} = f^Y_{ij}\mathcal{G} \), we note that \( R\pi_*f^Y_{ij}\mathcal{G} \in \mathcal{w} D^{>d^X_i} \).

Let \( S^Y_i := \bigcup_j S^Y_{ij} = \pi^{-1} S^X_i \). We will use an induction on the number of \( j \) to show that \( f^Y_i R\pi_*\mathcal{G} \in \mathcal{w} D^{>d^X_i} \). If \( j = 1 \), we have that \( R\pi_* f^Y_{1j}\mathcal{G} = f^Y_i R\pi_*\mathcal{G} \) (by base change), and this is in \( \mathcal{w} D^{>d^X_i} \) by the above.

If \( j > 1 \), let \( Z \xleftarrow{i} S^Y_{ij} \xrightarrow{j} U \) denote the closed stratum and its open complement (that is \( Z = S^Y_{ij} \) with \( S^Y_{ij} \) closed in \( S^Y_i \), and \( U = S^Y_i - Z \)). Let \( \mathcal{G}' = f^Y_i\mathcal{G} \in \mathcal{w} D^{>\text{Id}} \). Then we have a triangle:

\[ R\pi_* i^* f^Y_i \mathcal{G}' \to R\pi_* \mathcal{G}' \to R\pi_* j^* f^Y_i \mathcal{G}' \to \]

applying \( R\pi_* \) to the triangle for open and closed immersions. Now the first term is in \( \mathcal{w} D^{>d^X_i} \), because it is a projection from one stratum and so is the last term, by the induction hypothesis. Hence the fact that middle term is also contained \( \mathcal{w} D^{>\text{Id}} \) follows from the fact that for Morel’s t-structure these are triangulated subcategories. Thus, \( R\pi_* \mathcal{G}' \cong f^Y_i R\pi_*\mathcal{G} \in \mathcal{w} D^{>d^X_i} \) for all \( i \) as required and we are done.

It remains to construct the required stratification which is done in 3.1.12 below. \( \Box \)

3.1.11. Remark. In the argument above all that is needed is that \( D(n+1) - D(n) = 1 \) for the function \( D \). This gives for example, \( Rf_* (\mathcal{w} D^{\geq\text{Id}}) \subset \mathcal{w} D^{\geq\text{Id}} \) as well.

3.1.12. Lemma. Let \( \pi : Y \to X \) be any morphism. Then given stratifications \( S^{Y0}, S^{X0} \) of \( Y, X \), there are refinements \( S^Y, S^X \) such that:

- Each stratum of \( S^Y \) surjects onto a stratum of \( S^X \).
- Restricting \( \pi \) to a single stratum of \( S^Y \), each fibre is of same dimension.

Proof. Although this is standard we include a proof for the reader’s convenience. We reduce to the case where \( \pi \) is proper as follows: By Nagata’s compactification theorem (see [Con07]) we can extend the map \( \pi \) to a proper map \( \tilde{\pi} \) from a scheme \( \tilde{Y} \) such that \( Y \xhookrightarrow{\tilde{\pi}} \tilde{Y} \) is an open immersion. Then beginning with a stratification of \( \tilde{Y} \) obtained by adjoining the complement of \( Y \), and the given stratification on \( X \) we can recover the required stratification on \( Y \) by simply discarding those strata which do not lie in \( Y \).

For the proper case, we use induction on the dimension and number of irreducible components of \( Y, X \), the base case being trivial. In the following we will also denote by \( S^X, S^Y \) the stratification constructed at any intermediate stage.
Since $S_i^Y$ is locally closed, the image is constructible and hence the union of finitely many locally closed subsets. Hence refining the stratification of $X$ we can assume it is a union of finitely many strata of $X$, that is to say $S_i^Y \rightarrow \bigsqcup_j S_j^X$. Also let $\pi|S_i^Y = \pi_i$. Now we consider the stratification on $Y$ given by $\{\pi_i^{-1} S_j^X | i,j\}$ (these may not be smooth or irreducible). For this new stratification, we clearly have $S_i^Y \rightarrow S_{f(k)}^X$ for some mapping between the indexing sets $f$.

Now replacing $X$ by $S_k^X$, we can assume that $S_k^X = (X)$, $X$ is smooth irreducible, and $\pi$ restricted to each stratum of $Y$ is surjective. (Finally we will take union of stratifications constructed for each $S_k^X, \pi^{-1}_S X^k$.)

In characteristic 0, since our schemes are of finite type over the base field, there is an open set $U$ over which each of the maps $\pi|S_i^Y$ is smooth. Since $X$ is smooth, so are $U$ and $(\pi|S_i^Y)^{-1}U$. Hence we stratify $Y$ by the irreducible components of $(\pi|S_i^Y)^{-1}U$ and its complement. For the complement, $Z$, $\pi|(Y - \pi^{-1} Z)$ can be stratified satisfying the given conditions by induction on $\dim X$. In the end all we have to do is to take the union of strata.

In characteristic $p$, we can still assume that there is an open set over which fibres have constant dimension (follows from eg. [Har77, II, Ex.3.22]). Since our schemes are reduced, they are generically smooth, and we can choose an open set $V_i$ inside $(\pi|S_i^Y)^{-1}U$ which is smooth. The image $\pi(S_i^Y - V_i)$ is proper, closed subset for each $i$ (since the map is proper) and since $X$ is irreducible, so is their union. Now the same argument as above with $Z$ as union of the images gives what we want.

Now we make the key definitions:

3.1.13. Definition. Let $X$ be an equidimensional scheme of dimension $d_X$. Let $D$ be a non-decreasing step function. Further assume $0 \leq D \leq d_X$ with $D(d_X) = d_X$. For a smooth, open and dense subset $j : U \rightarrow X$ define:

$$EC_X^D := w_{\leq D} Rj_* Q_U$$

Using [Mor08 3.1.4] we see that there is a natural isomorphism $IC_X \cong w_{\leq d_X} Rj_* Q_U$ for any such $U \subset S$ and hence by 2.1.2 we also have:

$$EC_X^D \cong w_{\leq D} IC_X.$$

In particular $EC_X^D$ is independent of $U$.

3.1.14. Remark. Notice that if $D > d_X$, we would not be able to compare with the intersection complex and the definition may depend on $U$. If $D < 0$ then, since $D$ is a step function, $D(d_X) < d_X$ and hence $w_{\leq D} IC_X = w_{\leq D w_{\leq d_X}} IC_X = 0$. This explains the constraints.

The equidimensionality assumption can be avoided in one special case of interest:

3.1.15. Definition. For any $X$ we define

$$EC_X := w_{\leq d_X} Rj_* Q_U$$

for $j : U \rightarrow X$ smooth, open and dense. Thus for an equidimensional $X$ we have that $EC_X = EC_X^{Id}$. In any case it is independent of $U$ by the following lemma:
3.1.16. Lemma. Let $X$ be any arbitrary scheme and $j : U \hookrightarrow X$ be smooth open dense. Then:

$$w_{\leq 1d}Rj_!Q_U = w_{\leq 1d}IC_X$$

Proof. Let $X_i$ be the irreducible components of $X$, then by definition $IC_X = \bigoplus IC_{X_i}$. Now $Q_U = \bigoplus Rj_{i!}Q_{U_i}$ where $Q_{U_i}$ are constant sheaves supported on irreducible components $j_i : U_i \hookrightarrow U$ and $X_i = \bar{U}_i$. But then, using [Mor08, 3.1.4] and 2.1.2 we see that:

$$w_{\leq 1d}Rj_{i!}Q_{U_i} \cong w_{\leq 1d}w_{\leq \dim U_i}Rj_{i!}Q_{U_i} \cong w_{\leq 1d}IC_{X_i}$$

as required. \hfill \Box

3.1.17. Definition. Let $X$ be equidimensional and $D$ satisfying the constraints of 3.1.13 or $X$ be arbitrary and $D = \text{Id}$. Then we define:

$$EH^{D,j}(X) := \mathbb{H}^i(X, EC^D_X).$$

In particular,

$$EH^j(X) := EH^{\text{Id},j}(X) = \mathbb{H}^j(X, EC_X).$$

Notice that if $X$ is equidimensional $EH^{*,d_X}(X) = IH^*(X)$ (where $d_X$ stands for the constant function with value $\dim X$).

3.2. Properties of the weight-truncated complexes $EC^D_X$.

3.2.1. Proposition. Let $X$ be equidimensional and $D$ satisfying the constraints of 3.1.13. Define $E(i) = D(i) + 1$ if $D(i) < \dim X$ and $E(i) = \dim X$ otherwise. Then $E$ also satisfies the constraints of 3.1.13 and we have the equality:

$$EC^D_X \cong w_{\geq E}Rj_!Q_U$$

for $j : U \hookrightarrow X$ smooth open dense. If $X$ is arbitrary, then in any case:

$$EC_X \in wD^{\geq 1d}$$

Proof. The proof is immediate using [2.2.13]. For the second part note that if $U = \sqcup_i U_i$ in irreducible components, $EC_x = \bigoplus w_{\leq 1d}Rj_{i!}Q_{U_i}$ and we can work with one $U_i$ at a time. \hfill \Box

3.2.2. Proposition. Let $X$ be equidimensional and $D$ satisfying the constraints of 3.1.13 or $X$ be arbitrary and $D = \text{Id}$. Then:

1. The natural map $EC^D_X \rightarrow IC_X$ is an isomorphism for $D = \dim X$, the constant function.
2. There is a natural map $Q_X \rightarrow EC^D_X$ which is an isomorphism if $X$ is smooth.
3. Let $j : U \hookrightarrow X$ be an open dense immersion, then $EC^D_X \cong w_{\leq D}Rj_!EC^D_Y$. In particular $j^*EC^D_X \cong EC^D_Y$.
4. Let $\pi : Y \rightarrow X$ be a smooth map. Then, there is a natural isomorphism $\pi^*EC_X \cong EC_Y$.

Proof.

1. This is [Mor08, 3.1.4].
2. Consider any smooth stratification $S = (S_0, ..., S_r)$. Now $Q_X|_{S_i} \cong Q_{S_i} = (Q_{S_i}[\dim S_i])[\dim S_i]$ and hence is in $wD^{\leq \dim S_i}$ since for $S_i$ smooth $Q_{S_i}[\dim S_i]$ is perverse of weight $S_i$. Hence $Q_X \in wD^{\leq 1d} \subset wD^{\leq D}$.

We now obtain the required map by applying $w_{\leq D}$ on the adjunction map $Q_X \rightarrow Rj_!Q_U$ for a smooth open dense $U$. In case $X$ is smooth, we can choose $U = X$ and the isomorphism is immediate.
3.2.4. Proof. (1) Let us fix a stratification \( X \). The second term defines \( EC_X^D \) and it is enough to show that the third term vanishes. Choose a stratification of \( X \) adapted to \( R_j, w_D R_j^\ast Q_V \) such that \( V \) is a union of strata, and we immediately conclude that \( R_j, w_D R_j^\ast Q_V \subseteq w_D^\ast \) completing the argument.

(4) Let \( j : U \to X \) be open immersion with \( U \) smooth dense. Then if \( \tilde{U} = \pi^{-1} U \), \( \tilde{U} \) is also smooth (since \( \pi \) is) denote. Now consider the triangle:

\[
EC_X \to R_j^\ast Q_U \to w_{>1d} R_j^\ast Q_U \to
\]

Pulling back using \( \pi \) and using smooth base change, we get a triangle of complexes over \( Y \):

\[
\pi^\ast(EC_X) \to R_j^\ast Q_U \to \pi^\ast(w_{>1d} R_j^\ast Q_U) \to
\]

Now using 3.1.9 we see that the first term is in \( w_D \leq \text{Id} \) and last in \( w_D > \text{Id} \), and hence we must have the equality

\[
\pi^\ast(EC_X) \equiv w_{\leq 1d} R_j^\ast Q_U = EC_Y.
\]

\[\Box\]

3.2.3. Lemma. Let \( \mathcal{F} \in D^b(X), \mathcal{G} \in D^b(Y) \).

(1) Let \( A, B, C \) be monotone step functions such that

\[
A(i) + B(j) \leq C(i + j) \text{ for all } 0 \leq i \leq \text{dim} X, 0 \leq j \leq \text{dim} Y
\]

then if \( \mathcal{F} \in w_{D \leq A}(X), \mathcal{G} \in w_{D \leq B}(Y) \), then \( \mathcal{F} \boxtimes \mathcal{G} \in w_{D \leq C}(X \times Y) \).

Dually, if \( C \) is such that

\[
A(i) + B(j) \geq C(i + j) \text{ for all } 0 \leq i \leq \text{dim} X, 0 \leq j \leq \text{dim} Y
\]

then if \( \mathcal{F} \in w_{D \geq A}(X), \mathcal{G} \in w_{D \geq B}(Y) \), then \( \mathcal{F} \boxtimes \mathcal{G} \in w_{D \geq C}(X \times Y) \).

(2) Assume \( X = Y \). Let \( A, B, C \) be monotone step functions such that

\[
A(i) + B(i) \leq C(i) + i \text{ for all } 0 \leq i \leq \text{dim} X
\]

then if \( \mathcal{F} \in w_{D \leq A}, \mathcal{G} \in w_{D \leq B} \), then \( \mathcal{F} \boxtimes \mathcal{G} \in w_{D \leq C} \).

Proof. (1) Let us fix a stratification \( S \) adapted to \( \mathcal{F} \), and \( T \) adapted to \( \mathcal{G} \). Then stratification \( S \times T := \{ S_i \times T_j \mid i, j \} \) is adapted to \( \mathcal{F} \boxtimes \mathcal{G} \). Now \( f^\ast j^\ast \mathcal{F} \boxtimes f^\ast j^\ast \mathcal{G} \) is in \( w_{D \leq A(\text{dim} S_i) + B(\text{dim} T_j)} \) by 2.2.7. Therefore it is enough to show that \( A(\text{dim} S_i) + B(\text{dim} T_j) \leq C(\text{dim} S_i + \text{dim} T_j) \). But \( \text{dim} S_i \times T_j = \text{dim} S_i + \text{dim} T_j \), and the required inequality \( C(\text{dim} S_i + \text{dim} T_j) \geq A(\text{dim} S_i) + B(\text{dim} T_j) \) holds by assumption.

(2) Let us fix a stratification \( S \) adapted to \( \mathcal{F}, \mathcal{G}, \mathcal{F}_L \boxtimes \mathcal{G} \). Now the claim follows from 2.2.8

\[\Box\]

3.2.4. Corollary. Let \( X \) be equidimensional and \( D \) satisfying the constraints of 3.1.13 or \( X \) be arbitrary and \( D = \text{Id} \). Then:

(1) For any scheme \( Y \), \( EC_{X \times Y} \equiv EC_X \boxtimes EC_Y \).
There is a natural map \( \phi_D : EC_X \otimes EC^D_X \rightarrow EC^D_X \) such that we have a commutative diagram:

\[
\begin{array}{ccc}
Q_X \otimes Q_X & \xrightarrow{=} & Q_X \\
\downarrow & & \downarrow \\
Q_X \otimes EC^D_X & \xrightarrow{=} & EC^D_X \\
\downarrow & & \downarrow \\
EC_X \otimes EC^D_X & \xrightarrow{\phi_D} & EC^D_X
\end{array}
\]

We have an isomorphism of the compositions

\[
\phi_D \circ (1 \otimes \phi_D) \cong \phi_D \circ (\phi_{\text{Id}} \otimes 1)
\]

(1) Fix \( F = R_j^*Q_U \) and \( G = R_j^*Q_V \) for \( U \subset X \) and \( V \subset Y \) smooth open dense. Consider the triangle:

\[
w_{\leq \text{Id}} F \boxtimes w_{\leq \text{Id}} G \rightarrow w_{\leq \text{Id}} F \boxtimes G \rightarrow w_{\leq \text{Id}} F \boxtimes w_{> \text{Id}} G
\]

The first term is in \( w^{D \leq \text{Id}} \) by 3.2.3 while the last term is in \( w^{D > \text{Id}} \) using the same lemma since \( EC_X \in w^{D \geq \text{Id}} \) using 3.2.1. It follows that \( w_{\leq \text{Id}} F \boxtimes w_{\leq \text{Id}} G \cong w_{\leq \text{Id}}(w_{\leq \text{Id}} F \boxtimes G) \). Now consider the triangle

\[
w_{\leq \text{Id}} F \boxtimes G \rightarrow F \boxtimes G \rightarrow w_{> \text{Id}} F \boxtimes G
\]

Now if \( i : Z \rightarrow Y \) denotes the complement of \( V \), \( i^!G = 0 \), while \( j^!G = Q_V \in w^{D \geq \text{dim} V} \). Hence \( G \in w^{D \geq \text{Id}} \). Therefore, as before, using 3.2.3 \( w_{> \text{Id}} F \boxtimes G \in w^{D > \text{Id}} \). Consequently, we have that

\[ EC_X \boxtimes EC_Y \cong w_{\leq \text{Id}}(w_{\leq \text{Id}} F \boxtimes G) \cong w_{\leq \text{Id}}(F \boxtimes G) \cong EC_{X \times Y} \]

the identifications with \( EC_- \) being by definition.

(2) Let \( j : U \rightarrow X \) be smooth open dense subset of \( X \). Consider the natural morphism:

\[
w_{\leq \text{Id}} R_j^*Q_U \otimes w_{\leq D} R_j^*Q_U \rightarrow R_j^*Q_U \otimes R_j^*Q_U \rightarrow R_j^*Q_U
\]

where \( f \) is obtained by functoriality of \( \otimes \), and \( g \) is obtained via adjunction of natural morphism \( j^*(R_j^*Q_U \otimes R_j^*Q_U) \cong j^*R_j^*Q_U \otimes j^*R_j^*Q_U \cong Q_U \xrightarrow{\text{Id}} Q_U \). Now by 3.2.3 we have that \( w_{\leq \text{Id}} R_j^*Q_U \otimes w_{\leq D} R_j^*Q_U \in w^{D \leq D} \), and hence the composition \( gf \) factors through a natural map:

\[
w_{\leq \text{Id}} R_j^*Q_U \otimes w_{\leq D} R_j^*Q_U \rightarrow w_{\leq D} R_j^*Q_U
\]

and we constructed the required morphism.
To see the commutativity properties, note that in the natural diagram:

\[
\begin{array}{ccc}
Q_X \otimes Q_X & \overset{m}{\longrightarrow} & Q_X \\
\downarrow \text{Id} \otimes a_D & & \downarrow a_D \\
Q_X \otimes EC_X^D & \overset{n}{\longrightarrow} & EC_X^D \\
\downarrow a \otimes \text{Id} & & \downarrow \\
EC_X \otimes EC_X^D & \overset{\phi_D}{\longrightarrow} & EC_X^D \\
\downarrow f & & \downarrow \beta \\
R_{j*}Q_U \otimes R_{j*}Q_U & \overset{\phi}{\longrightarrow} & R_{j*}Q_U
\end{array}
\]

the outer square commutes. But cone of $\beta$ lies in $wD^{>D}$ while $Q_X \in wD^{\leq \text{Id}} \subset wD^{\leq D}$, hence we must have that $\phi_D \circ (a \otimes \text{Id}) \circ (\text{Id} \otimes a_D) = a_D \circ m$ as required.

(3) Tracing the definitions, both compositions are also obtained in the following way – consider the natural map which is composition of:

\[
w_{\leq \text{Id}}R_{j*}Q_U \otimes w_{\leq \text{Id}}R_{j*}Q_U \overset{L}{\longrightarrow} w_{\leq D}R_{j*}Q_U \overset{L}{\longrightarrow} w_{\leq D}R_{j*}Q_U \overset{L}{\longrightarrow} R_{j*}Q_U \overset{L}{\longrightarrow} R_{j*}Q_U
\]

(first map coming from truncation, and second adjunction as before). Since the first term lies in $wD^{\leq D}$, this factors through a map

\[
w_{\leq \text{Id}}R_{j*}Q_U \otimes w_{\leq \text{Id}}R_{j*}Q_U \overset{L}{\longrightarrow} w_{\leq D}R_{j*}Q_U \overset{L}{\longrightarrow} w_{\leq D}R_{j*}Q_U
\]

which is the required composition.

\[\square\]

3.2.5. **Lemma.** Let $X$ be a scheme and $D$ be a monotone step function. Then $w_{\leq D}$ is left exact (with respect to the standard $t$-structure).

**Proof.** Let $\mathcal{F} \in D^\geq 0$. We fix a stratification $S$ adapted to $X$ and hence we are reduced to computing $\mathcal{F}^{\leq (d_0, d_1, \ldots, d_r) \mathcal{F}}$. Now use 2.1.4 and 2.2.9. \[\square\]

3.2.6. **Lemma.** Let $X$ be a scheme and $D$ be a non-decreasing step function. Then $w_{\leq D}$ is perverse right exact.

**Proof.** Let $\mathcal{F} \in \mathcal{D}^{<,0}$. Let us first fix a stratification $S$ adapted to $\mathcal{F}$. Thus we have to compute $\mathcal{F}^{\leq (d_0, d_1, \ldots, d_r) \mathcal{F}}$, where $d_i \leq d_{i-1}$. We omit $S$ from the notation for simplicity. We have:

\[
w_{\leq (d_0, d_1, \ldots, d_r) \mathcal{F}} = w_{\leq (d_0, d_1, \ldots, d_r)} w_{\leq (d_0, d_1, \ldots, d_{r-1}, d_{r-1})} \cdots w_{\leq (d_0, d_1, \ldots, d_1)} w_{\leq (d_0, d_0, \ldots, d_0)} \mathcal{F}
\]

\[
= w_{\leq (+\infty, +\infty, \ldots, d_r)} w_{\leq (+\infty, +\infty, \ldots, d_{r-1}, d_{r-1})} \cdots w_{\leq (+\infty, d_1, \ldots, d_1)} w_{\leq (d_0, d_0, \ldots, d_0)} \mathcal{F}
\]

(The first equality holds using 2.1.2. The second equality holds using further that $w_{\leq (d_0, \ldots, d_r, +\infty, \ldots)}$ is the identity on objects in $wD^{\leq (d_0, \ldots, d_r)}$, as is seen using 2.1.2. Thus noting that $w_{\leq (+\infty, \ldots, +\infty, d, \ldots, d)} = w_{\leq (+\infty, d)}$ for an appropriate re-stratification, the result follows from 2.2.10. \[\square\]
3.2.7. Corollary. If $X$ is equidimensional and $D$ satisfies the constraints of [3.1.13] or if $X$ is arbitrary and $D = \text{Id}$, then
\[ EC^D_X \in \mathbb{p}D^{\leq \dim X}(X) \text{ and } EC^D_X \in D^{\geq 0}(X) \]

Proof. Use [3.2.5] and [3.2.6] on the two isomorphisms [3.2.8].

3.2.8. Proposition. For $X$ equidimensional of dimension $d_X$ and $D$ satisfying the constraints of [3.1.13] we have an isomorphism
\[ EH^{D,2d_X}(X) \cong IH^{2d_X}(X) \]
and a surjection
\[ EH^{D,2d_X-1}(X) \to IH^{2d_X-1}(X). \]

Proof. Consider the triangle:
\[ EC^D_X \to IC_X \to w_D IC_X. \]
Since for some open dense $U$, $j^* IC_X = \mathbb{Q}_U \in w D^{\leq D}$, by [2.1.4] and [2.1.5] we have $w_D IC_X = i_* w_D j^* IC_X$. (Here $j : U \to X$ and $i : X-U \to X$.)

Now consider the induced long exact sequence:
\[ \to p H^i(EC^D_X) \to p H^i(IC_X) \to p H^i(w_D IC_X) \to p H^{i+1}(EC^D_X) \to \]
Since $EC^D_X \in \mathbb{p}D^{\leq d_X}$ by [3.2.7], therefore $p H^i(w_D IC_X) = 0$ for $i \geq d_X + 1$.
Further for $i = d_X$ we get a surjection:
\[ p H^{d_X}(IC_X) \to p H^{d_X}(w_D IC_X) \to 0 \]
But $p H^{d_X}(w_D IC_X) = i_* p H^{d_X}(w_D j^* IC_X)$ has strictly proper support while $p H^{d_X}(IC_X) = IC_X[d_X]$ has full support. Hence $p H^{d_X}(w_D IC_X)$ vanishes as well and $w_D IC_X \in \mathbb{p}D^{\leq d_X-1}$.

Since $w_D IC_X$ is supported on a strictly proper subvariety (of dimension $\leq d_X - 1$) we have $\mathbb{H}^i(X, w_D IC_X) = 0$ for $i > 2d_X - 2$. Now the long exact sequence of cohomology for the above triangle yields:
\[ \to \mathbb{H}^{i-1}(X, w_D IC_X) \to EH^{D,i}(X) \to IH^i(X) \to \mathbb{H}^i(X, w_D IC_X) \to \]
For $i = 2d_X$ we see that $EH^{D,2d_X}(X) \cong IH^{2d_X}(X)$ and for $i = 2d_X - 1$ we get a surjection $EH^{D,2d_X-1}(X) \to IH^{2d_X-1}(X)$ as required. \hfill \Box

4. Pullbacks and other properties

4.1. Lemma. Let $\pi : Y \to X$ be any morphism, and $j_X : U \to X$ be open. Let $j_Y : U_Y = \pi^{-1}U \to Y$ denote the base change. If $\mathcal{G}$ is in $w D^{\geq \dim Y}(Y)$ and $U_Y \subset Y$ is dense, the following natural map is an isomorphism:
\[ w_{\leq \id} R\pi_* \mathcal{G} \cong w_{\leq \id} Rj_{Y*} j^* X R\pi_* \mathcal{G} \equiv w_{\leq \id} Rj_{X*} R\pi_* j^*_Y \mathcal{G} \]
(the second isomorphism is due to smooth base change, here $\pi$ also denotes $\pi|_U$).
Proof. Let $i_Y : Z_Y \hookrightarrow Y$ denote the inclusion of the closed complement. Consider the triangle:

$$i_Y \circ i_Y^\ast \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow Rj_Y \circ j_Y^\ast \mathcal{G} \longrightarrow \text{If } \mathcal{G} \in wD^{\geq \dim Y}, i_Y^\ast \mathcal{G} \in wD^{\geq \dim Y} \subset wD^{> \text{Id}} \text{ (because } \dim Z_Y < \dim Y \text{ as } U_Y \text{ is dense). Hence } w_{\leq \text{Id}}R\pi_Y^\ast \mathcal{G} \cong w_{\leq \text{Id}}Rj_Y^\ast j_Y^\ast \mathcal{G} \cong w_{\leq \text{Id}}Rj_X^\ast R\pi_X^\ast j_Y^\ast \mathcal{G}.$$ 

4.1.2. Proposition. Let $\pi : Y \rightarrow X$ be generically proper with each generic fibre geometrically connected (i.e. assume that there is an open dense $U \subset Y$ such that $\pi : U \rightarrow \pi(U)$ is proper with geometrically connected fibres). Also assume that $Y$ is smooth over $\text{Spec } k$ and $\pi$ is such that each irreducible component of $Y$ dominates some component of $X$. Then $EC_X \cong w_{\leq \text{Id}}R\pi_Y^\ast Q_Y$.

Proof. For simplicity assume $X$ is irreducible, the proof remains similar otherwise. By the previous lemma with $\mathcal{G} = Q_Y$,

$$w_{\leq \text{Id}}R\pi_Y^\ast Q_Y \cong w_{\leq \text{Id}}Rj_Y^\ast R\pi_X^\ast Q_{\pi_X^{-1}U}.$$ 

Now by 3.1.10, right hand side is also same as $w_{\leq \text{Id}}Rj_Y^\ast w_{\leq \text{Id}}R\pi_X^\ast Q_{\pi_X^{-1}U}$. But by choosing $U$ small enough, $R\pi_X^\ast Q_{\pi_X^{-1}U}$ is lisse on $U$, and therefore we have an equality $w_{\leq \text{Id}}R\pi_X^\ast Q_{\pi_X^{-1}U} = w_{\leq \dim U}R\pi_Y^\ast Q_{\pi_Y^{-1}U}$. But by the decomposition theorem on $U$ we see immediately that $w_{\leq \dim U}R\pi_Y^\ast Q_{\pi_Y^{-1}U} = R^0\pi_Y^\ast Q_{\pi_X^{-1}U} = Q_U$ (the last equality because the generic fibre is geometrically connected). Notice that the identification only uses the natural morphism $\pi_X^\ast Q_{\pi_X^{-1}U} \rightarrow R\pi_Y^\ast Q_{\pi_Y^{-1}U}$ and hence is canonical. Now the result follows immediately from definition of $EC_X$.  

4.1.3. Remark. If the generic fibre is not geometrically connected, assuming that there is an open dense set over which $\pi$ is proper, we have that $Q_U$ is a summand of $\pi_X^\ast Q_{\pi_X^{-1}U}$ for small enough $U$. Then $EC_X$ is a summand of $w_{\leq \text{Id}}R\pi_Y^\ast Q_Y$.

4.1.4. Proposition. For any morphism $f : Y \rightarrow X$ such that the image of each irreducible component of $Y$ meets the smooth locus of $X$, there is an induced morphism $f^\# : EC_X \rightarrow Rf^\ast EC_Y$.

If $g : Z \rightarrow Y$ is another morphism such that both $g$ and $f \circ g$ satisfy this condition (i.e. the image of each irreducible component of the domain meets the codomain), then there is a natural identification $(f \circ g)^\# = Rf^\ast g^\# \circ f^\#$.

Proof. Consider the open immersion $\pi_X : \tilde{X} \rightarrow X$ where $\tilde{X}$ is the smooth locus of $X$. Let $\tilde{Y}$ be the smooth locus of the fibred product $\tilde{X} \times_X Y$ (which is nonempty since the image of $Y$ meets the smooth locus of $X$), and let the composite $\tilde{Y} \rightarrow Y$ be denoted $\pi_Y$.

Now consider the following diagram:

$$EC_X \xrightarrow{\phi} w_{\leq \text{Id}}R\pi_X^\ast Q_X \xrightarrow{\phi} Rf^\ast w_{\leq \text{Id}}R\pi_Y^\ast Q_Y \xrightarrow{\phi} Rf^\ast EC_Y$$

We denote the composition $EC_X \rightarrow Rf^\ast EC_Y$ by $f^\#$ (Here the dotted arrow comes from the fact that Hom$(wD^{\leq \text{Id}}, wD^{> \text{Id}}) = 0$ and using the long exact sequence of Hom’s along with invariance of Morel’s $t$-structure under shifts.)
To see that \((f \circ g)^\# = Rf_*g^\# \circ f^\#\), one only needs to arrange a tower \(\tilde{Z} \to \tilde{Y} \to \tilde{X}\), here \(\tilde{Z}\) is smooth locus of the fibre product \(\tilde{Y} \times_Y \tilde{Z}\). It is enough to show that the fibre product is nonempty. Let \(W \subset Y\) be the preimage of the smooth locus of \(X\) under \(f\) and let \(U \subset Y\) be the smooth locus of \(Y\). Then \(\tilde{Y} = U \cap W\). If the image of some irreducible component does not meet \(\tilde{Y}\), since it meets \(U - W\) and \(W - U\) (because of our assumptions on \(f \circ g\) and \(g\) respectively), it is not irreducible, a contradiction. \(\square\)

4.1.5. Proposition. Let \(f : Y \to X\) be such that the image of each irreducible component of \(Y\) meets the smooth locus of \(X\). Then \(f^\#\) above is a morphism of ring objects. The natural diagram

\[
\begin{array}{c}
Q_X \longrightarrow Rf_*Q_Y \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
EC_X \overset{f^\#}{\longrightarrow} Rf_*EC_Y
\end{array}
\]

commutes.

Proof. Take \(\pi_X : \tilde{X} \to X\) to be the smooth locus of \(X\) and \(\pi_Y : \tilde{Y} \to Y\) to be the smooth locus of the fibre product. By \[3.2.2\] the natural map \(Q_X \to R\pi_X_*Q_{\tilde{X}}\) factors through a map \(g_X : Q_X \to EC_X\) (similarly for \(Y\)), and hence we get a diagram:

\[
\begin{array}{c}
Q_X \quad \overset{a}{\longrightarrow} \quad Rf_*Q_Y \\
\downarrow g_X \quad \quad \quad \quad \quad \quad \quad \quad \downarrow Rf_*g_Y \\
EC_X \overset{f^\#}{\longrightarrow} Rf_*EC_Y \\
\downarrow b \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
R\pi_X_*Q_{\tilde{X}} \overset{R\pi_X*(adj.)}{\longrightarrow} Rf_*R\pi_Y_*Q_{\tilde{Y}}
\end{array}
\]

where the topmost arrow is the natural adjunction.

Since the outer square and the bottom square commute, we see that \(b \circ f^\# \circ g_X = b \circ Rf_*g_Y \circ a\). But \(\text{Cone}(b) = Rf_*w^{>1d}R\pi_YQ_{\tilde{Y}} \in wD^{>1d}\), while \(Q_X \in wD^{\leq 1d}\). Hence we must have \(f^\# \circ g_X = Rf_*g_Y \circ a\) as required.

To see that \(f^\#\) is a morphism of ring objects, consider the natural diagram:

\[
\begin{array}{c}
EC_X \otimes EC_X \quad \overset{L}{\longrightarrow} \quad R\pi_X_*Q_{\tilde{X}} \otimes R\pi_X_*Q_{\tilde{X}} \\
\downarrow f^\# \otimes f^\# \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
Rf_*EC_Y \otimes Rf_*EC_Y \quad \overset{L}{\longrightarrow} \quad Rf_*R\pi_Y_*Q_{\tilde{Y}} \otimes Rf_*R\pi_Y_*Q_{\tilde{Y}}
\end{array}
\]

The bottom and top faces commute by definition of \(f^\#\). The front and back faces commute as in the proof of \[3.2.4\]. The rightmost face commutes because ordinary cohomology is a
morphism of ring objects. Hence we have that the left face commutes after composing with α, that is:

\[ \alpha \circ f^# \circ m_X = \alpha \circ m_Y \circ (f^# \otimes f^#). \]

Now notice that the cone of α is \( R_f w_{>1d} R\pi_* \mathbb{Q}_Y \in \mathcal{W}D^{>1d} \) while \( EC_X \otimes EC_X \in \mathcal{W}D^{\leq1d} \). Hence we must already have \( f^# \circ m_X = m_Y \circ (f^# \otimes f^#) \) as required. \( \square \)

4.1.6. **Remark.** It is possible to define canonical pullbacks for arbitrary morphisms, however they do not seem to satisfy the composition law as above nor do they seem to be morphisms of ring objects in general. The key observation one uses for the construction of general pullbacks is that even if the generic fibre of \( \pi : Y \to X \) is not geometrically connected, but \( \pi \) is generically proper, dominant and \( Y \) smooth, \( EC_X \) is canonically a summand of \( w_{\leq1d} R\pi_* \mathbb{Q}_Y \). See [Vai12] for more details.

4.2. **Other properties.** The following theorem summarizes the properties of the cohomology groups \( EH^{D,*}(X) \):

4.2.1. **Theorem.** Let \( X \) be an equidimensional variety and \( D \) a monotone non-decreasing step function satisfying the constraints of 3.1.13 or \( X \) be arbitrary and \( D = \text{Id} \). The graded group \( EH^{D,*}(X) \) satisfies:

1. If \( X \) is of dimension \( d \) then \( EH^{D,i} \not= 0 \) only if \( i \in [0,2d] \). If \( \nu : \hat{X} \to X \) is the normalization then \( EH^{D,*}(X) \cong EH^{D,*}(\hat{X}) \). If \( X \) is smooth then \( H^*(X) \cong EH^{D,*}(X) \).

For \( D = \dim X \), the constant function, \( EH^{\dim X,i}(X) \cong IH^i(X) \).

2. \( EH^*(X) \) is a graded-commutative ring. The natural map \( H^*(X) \to EH^*(X) \) is a ring map.

3. \( EH^*(X) \) has ring pullbacks for morphisms with the image of each component meeting the smooth locus of the codomain.

4. \( EH^*(X \times Y) \cong EH^*(X) \otimes EH^*(Y) \).

5. \( EH^{D,*} \) live between the usual cohomology and the intersection cohomology, that is for all \( i \) there are natural maps:

\[ H^i(X) \to EH^{D,i}(X) \to IH^i(X) \]

6. If \( d = \dim X \), the second map induces:

\[ EH^{D,2d} \to IH^{2d} \]

\[ EH^{D,2d-1} \to IH^{2d-1} \]

7. \( EH^{D,*}(X) \) are graded modules over \( EH^*(X) \), and in particular \( IH^*(X) \) is a graded module over \( EH^*(X) \). Furthermore, the natural action of the ring \( H^*(X) \) on \( EH^{D,*}(X) \) factors through that of \( EH^*(X) \).

**Proof.** (1) The bounds follow from 3.2.7 since for \( F \in D^{>0} \) we have \( \mathbb{H}^i(X,F) = 0 \) for \( i < 0 \), and if \( F \in D^{\geq \dim X} \) we have \( \mathbb{H}^i(X,F) = 0 \) for \( i > 2 \dim X \).

For the second statement, let \( j : U \inj \hat{X} \) denote a dense open smooth subset of \( \hat{X} \) mapping isomorphically into \( X \). We let \( j = \nu \circ j : U \inj X \) denote the inclusion. But then, 3.1.8 implies that

\[ R\nu_* EC_X = R\nu_* w_{\leq1d} Rj_* \mathbb{Q}_U = w_{\leq1d} Rj_* \mathbb{Q}_U = EC_X \]

which is what we require. The last two statements follow from 3.2.2.
(2) This follows from 3.2.4 for \( D = \text{Id} \). It is clear that we have a commutative diagram:

\[
\begin{align*}
\mathbb{Q} \otimes \mathbb{Q} & \longrightarrow \mathbb{Q} \\
\mathbb{Q} \otimes \mathbb{Q} & \longrightarrow \mathbb{Q}
\end{align*}
\]

which tells us that the natural map \( H^i(X) \rightarrow EH^i(X) \) is a ring map.

(3) This follows from 4.1.4 and 4.1.5.

(4) This follows from 3.2.4.

(5) The second map comes from the map \( EC_X \rightarrow IC_X \) of 3.1.16, while the first map comes from the map \( \mathbb{Q} \rightarrow EC_X \) of 3.2.2.

(6) This follows from 3.2.8.

(7) This again follows from 3.2.4 taking \( D \) other than \( \text{Id} \). We have a commutative diagram:

\[
\begin{align*}
\mathbb{Q} \otimes EC_X & \longrightarrow EC_X \\
\mathbb{Q} \otimes EC_X & \longrightarrow EC_X
\end{align*}
\]

which tells us that the action of \( H^*(X) \) on \( EH^D, X \) factors through the action of \( EH^D_X \), in particular for \( D = \text{dim} X \) we recover the statement on intersection cohomology.

\[\square\]

4.3. Weights. In this subsection we prove properties of the weight filtration on \( EH^*(X) \) (including (viii) and (ix) of the introduction).

Consider first the special case when \( X \) is proper.

4.3.1. Proposition. If \( X \) is proper then \( \text{Gr}^W_i EH^i(X) = 0 \) for \( j > i \) and the map \( EH^*(X) \rightarrow IH^*(X) \) induces a natural isomorphism

\[
\text{Gr}^W_i EH^i(X) = \text{im}(EH^i(X) \rightarrow IH^i(X)).
\]

Proof. For the first assertion it is enough to show that the complex \( EC_X \) has weights \( \leq 0 \). Let \( X = \sqcup_i S_i \) be a stratification with \( U = S_0 \) open and dense, for which \( Rj_* \mathbb{Q}_U, j\mathbb{Q}_U, \) and \( IC_X \) are constructible. Let \( d_i := \text{dim} S_i \) and let \( i_k : S_k \hookrightarrow X \) be the inclusions. Then \( EC_X = w_{\leq (d_0,\ldots,d_r)} Rj_* \mathbb{Q}_U = w_{\leq (d_0,\ldots,d_r)} IC_X \). Since \( d_0 \geq \cdots \geq d_r \), one has

\[
w_{\leq (d_0,\ldots,d_r)} = w_{\leq (+\infty,\ldots,+\infty,d_r)} \circ \cdots \circ w_{\leq (+\infty,d_1,\ldots,d_1)} \circ w_{\leq (d_0,\ldots,d_0)}
\]

(as in the proof of 3.2.6). Each term on the right is of the form \( w_{\leq (\infty,d)} \) for a stratification with two strata, so respects the condition of having weights \( \leq 0 \) by 2.2.11. Since \( IC_X \) is pure of weight zero, \( EC_X \) has weights \( \leq 0 \).

For the second assertion we use the triangle

\[
EC_X \rightarrow IC_X \rightarrow w_{>(d_0,\ldots,d_r)} IC_X \rightarrow.
\]

By the fact that \( EC_X \) has weights \( \leq 0 \) and the hypercohomology long exact sequence of this triangle, it is enough to show that \( K = w_{>(d_0,\ldots,d_r)} IC_X \) has weights \( \leq 0 \). For this we will use the pointwise criterion for weights: \( K \in D^b(X) \) has weights \( \leq 0 \) if, and only if, for every closed point \( i_x : \{x\} \hookrightarrow X, H^i(i_x^* K) \) has weights \( \leq i \). If \( x \in U = X_0 \) this obviously
holds, so assume \( x \in S_k \) for \( k \geq 1 \). Then \( i_k^* EC_X = i_k^* w_{\leq (d_0, \ldots, d_r)} Rj_* Q/U \) belongs to \( w D^{\leq d_k} \), i.e. \( pH^i(i_k^* EC_X) \) has weights \( \leq d_k \) for all \( i \). Since \( EC_X \) is constructible for the stratification, 
\[
pH^i(i_k^* EC_X) = H^{i-d_k}(i_k^* EC_X)[d_k],
\]
so that \( H^i(i_k^* EC_X) \) has weights \( \leq d_k - d_k = 0 \) for all \( i \). In the long exact sequence 
\[
\cdots \to H^i(i_k^* IC_X) \to H^i(i_k^* K) \to H^{i+1}(i_k^* EC_X) \to \cdots
\]
the first group has weights \( \leq i \) (because \( IC_X \) has weight 0) and the last group has weights \( \leq 0 \). So \( H^i(i_k^* K) \) has weights \( \leq i \) for all \( i \), proving that \( K = w_{> (d_0, \ldots, d_r)} IC_X \) is of weight \( \leq 0 \).

4.3.2. Remark. The proposition implies the following weak version of the hard Lefschetz theorem: If \( e \in H^2(X) \) is the first Chern class of an ample line bundle then \( \cdot e : Gr^W EH^i(X) \to Gr^W_{i+2} EH^{i+2}(X) \) is injective if \( i < \text{dim } X \). This is used in [Nai10].

The bounds on the weights of \( EH^*(X) \) in the general case will follow from the following statement (which also contains another proof of the previous proposition).

4.3.3. Theorem. Let \( X \) be an irreducible scheme and let \( \pi : Y \to X \) be proper with \( Y \) smooth. Then:
\[
Gr^W_j \mathbb{H}^i(X, w_{\leq \text{id}} R\pi_* Q_Y) = 0 \text{ if } j < 2i - 2 \text{dim } Y \text{ or } j > 2i \text{ or } j > 2 \text{dim } Y
\]
If, in addition, \( X \) is proper then:
\[
Gr^W_j \mathbb{H}^i(X, w_{\leq \text{id}} R\pi_* Q_Y) = 0 \text{ if } j > i
\]
\[
Gr^W_i \mathbb{H}^i(X, w_{\leq \text{id}} R\pi_* Q_Y) \cong \text{im}(\mathbb{H}^i(X, w_{\leq \text{id}} R\pi_* Q_Y) \to \mathbb{H}^i(X, R\pi_* Q_Y) = H^i(Y))
\]
(Here the \( Gr^W_i \) are the graded with respect to the weight filtration.)

Proof. We will prove the fact using Noetherian induction on \( Y \). The proof of all the statements is very similar, and we prove them together.

Reduction to \( Y \) irreducible: \( Y = \sqcup Y_i \) with \( Y_i \) smooth. Then \( \text{dim } Y_i \leq \text{dim } Y \). Hence vanishings for \( j < 2i - 2 \text{dim } Y \) (resp. \( j > 2 \text{dim } Y \)) imply \( j < 2i - \text{dim } Y_i \) (resp. \( j > 2 \text{dim } Y_i \)) and we see that the statement for \( Y_i \) implies that for \( Y \).

Reduction to \( \pi \) surjective: Since \( \pi \) is proper, its image is closed. Let \( i : X' \to X \) be the closed image of \( \pi \) and \( \pi' : Y \to X' \) be the restriction of \( \pi \). Then \( \pi = i \pi' \). We also have:
\[
w_{\leq \text{id}} R\pi_* Q_Y \cong w_{\leq \text{id}} i_* R\pi'_* Q_{Y'} \cong i_* w_{\leq \text{id}} R\pi'_* Q_{Y'}
\]
But \( \mathbb{H}^i(X, i_* F) = \mathbb{H}^i(X', F) \) for any complex of sheaves, and hence we can replace \( \pi \) by \( \pi' \) and \( X \) by \( X' \), thereby assuming that \( \pi \) is surjective.

Base case of induction: Since \( Y \) is smooth of dimension zero and irreducible, \( Y \) is Spec \( k \). Since \( \pi \) is surjective, the underlying space of \( Y \) is a point as well. Now the claims are obvious.

Induction Step:

Case 1: \( \pi \) is cohomologically lisse outside an snc divisor: Assume first that there is some \( j : U \to X \) open immersion of a smooth set such that \( j^* R\pi_* Q_Y \) is lisse on \( U \), and if \( i : Z \to X \) denotes its closed complement, \( Y' := \pi^{-1} Z \) is a simple normal crossing divisor.

Step 1: Reducing to a condition on \( T = w_{> \text{id}} \tau \leq d_X R\pi_* Q_Y \):
Consider the triangle:
\[
\tau \leq d_X R\pi_* Q_Y \to R\pi_* Q_Y \to \tau > d_X R\pi_* Q_Y \to \cdots
\]
Now $Q_Y$ is a sheaf of weight 0. Since $\pi$ is proper, $p^jH^i(R\pi_*Q_Y)$ is of weight $i$. Hence $p_{\tau > d_X}R\pi_*Q_Y$ belongs to $wD^{>d_X}(X)$ and hence to $wD^{>1d}(X)$. Hence
\[ Hom(w_{\leq 1d}R\pi_*Q_Y, p_{\tau > d_X}R\pi_*Q_Y[i]) = 0 \]
for all $i$ and using the long exact sequence of $Hom$’s for the above triangle, we get that the natural map $w_{\leq 1d}R\pi_*Q_Y \to R\pi_*Q_Y$ uniquely lifts to a map $w_{\leq 1d}R\pi_*Q_Y \to p_{\tau \leq d_X}R\pi_*Q_Y$. We consider the octahedral diagram arising out of this situation:

(Here $T$ is defined by the left most triangle.) Now since $p_{\tau > d_X}R\pi_*Q_Y \in wD^{>1d}$ and hence applying $w_{\leq 1d}$ to the triangle above, we also have an identification,
\[ w_{\leq 1d}R\pi_*Q_Y \cong w_{\leq 1d}p_{\tau \leq d_X}R\pi_*Q_Y \]
and hence
\[ T \cong w_{\leq 1d}p_{\tau \leq d_X}R\pi_*Q_Y. \]

Consider the induced long exact sequence from top left triangle:
\[ \to \text{Gr}^W_{j}H^{i-1}(X, T) \to \text{Gr}^W_{j}H^{i}(X, w_{\leq 1d}R\pi_*Q_Y) \to \text{Gr}^W_{j}H^{i}(X, p_{\tau \leq d_X}R\pi_*Q_Y) \to \]

By the decomposition theorem, the last term is a summand of the $\text{Gr}^W_{j}H^{i}(X, R\pi_*Q_Y) = \text{Gr}^W_{j}H^i(Y)$ and hence vanishes for

(i) $j < 2i - 2 \dim Y$
(ii) $j > 2i$
(iii) $j > 2 \dim Y$
(iv) $j > i$ for $X$ (and hence $Y$) proper

by Deligne’s theory of weights. Hence it is enough to show that the first term $\text{Gr}^W_{j}H^{i-1}(X, T)$ vanishes for the same range of $j$, to obtain vanishing of the middle term.

Claim: If $X$ is proper, the vanishing of $\text{Gr}^W_{j}H^{i-1}(X, T)$ implies that $\text{Gr}^W_{j}H^{i}(X, w_{\leq 1d}R\pi_*Q_Y) = \text{im}(\text{Gr}^W_{i}(X, w_{\leq 1d}R\pi_*Q_Y) \to H^i(Y))$.

Proof of the claim: The vanishing implies an injection:
\[ \text{Gr}^W_{j}H^{i}(X, w_{\leq 1d}R\pi_*Q_Y) \hookrightarrow \text{Gr}^W_{j}H^{i}(X, p_{\tau \leq d_X}R\pi_*Q_Y) \]
The second term is a direct summand of $H^i(Y)$, and hence the morphism
\[ \text{Gr}^W_{j}H^{i}(X, w_{\leq 1d}R\pi_*Q_Y) \to \text{Gr}^W_{j}H^i(Y) = H^i(Y) \]
is injective. Since $H^i(Y)$ is pure of weight $i$, the image $\text{im}(\text{Gr}^W_{i}(X, w_{\leq 1d}R\pi_*Q_Y) \to H^i(Y))$ is pure of weight $i$, and hence a subquotient of the $i$-th graded. But by injectivity of the above map, it must be precisely the $i$-th graded, as required.

Step 2: Reduction to the divisor in the boundary
Since \( j : U \hookrightarrow X \) is an open immersion we have

\[
j^*(p_{\tau \leq d_X} R\pi_* Q_Y) \cong p_{\tau \leq d_X} R\pi_{U*} Q_{Y_U}
\]

where \( \pi_U = \pi|_U : Y_U = \pi^{-1}U \to U \). Since \( j^* R\pi_* Q_Y = R\pi_{U*} Q_{Y_U} \) is lisse, we further have

\[
p_{\tau \leq d_X} R\pi_{U*} Q_{Y_U} \cong \tau_{\leq 0} R\pi_{U*} Q_{Y_U}
\]

and this is pure of weight 0.

Now considering \( \text{Id} \) as the glued \( t \)-structure \([\text{Id}; \text{Id}]\) on strata \((U, Z)\), we have an isomorphism:

\[
T \cong w_{(\text{Id}, \text{Id})}(p_{\tau \leq d_X} R\pi_* Q_Y) \cong w_{(\text{Id}, \text{Id})}(p_{\tau \leq d_X} R\pi_* Q_Y)
\]

Now consider the triangle:

\[
j^!w_{\leq \text{Id}} j^*(p_{\tau \leq d_X} R\pi_* Q_Y) \to p_{\tau \leq d_X} R\pi_* Q_Y \to w_{(\text{Id}, \text{Id})}(p_{\tau \leq d_X} R\pi_* Q_Y)
\]

But, by the above, the first term is nothing but:

\[
j^!w_{\leq \text{Id}} j^*(p_{\tau \leq d_X} R\pi_* Q_Y) = j^!w_{\leq \text{Id}} R^0\pi_{U*} Q_{Y_U} \cong j^!R^0\pi_{U*} Q_{Y_U} \cong j^!j^*(p_{\tau \leq d_X} R\pi_* Q_Y)
\]

(note that \( R^0\pi_{U*} Q_{Y_U} = \tau_{\leq 0} \pi_{U*} Q_{Y_U} \)) and comparing with the localization triangle, we have an isomorphism:

\[
w_{(\text{Id}, \text{Id})}(p_{\tau \leq d_X} R\pi_* Q_Y) \cong \text{i}_*\text{i}^*(p_{\tau \leq d_X} R\pi_* Q_Y)
\]

Consequently, we also have an isomorphism:

\[
T \cong w_{\geq \text{Id}} p_{\tau \leq d_X} R\pi_* Q_Y \cong w_{\geq \text{Id}} \text{i}_*\text{i}^*(p_{\tau \leq d_X} R\pi_* Q_Y)
\]

But then by the decomposition theorem, the last term is a summand of \( w_{\geq \text{Id}} \text{i}_*\text{i}^*(R\pi_* Q_Y) = \text{i}_*w_{\geq \text{Id}} R\pi_* Q_{YZ} \) and it is enough to show that the graded

\[
Gr^W_{\mathbb{H}_{i-1}}(X, i_*w_{\geq \text{Id}}(R\pi_* Q_{YZ})) \cong Gr^W_{\mathbb{H}_{i+1}}(Z, w_{\geq \text{Id}}(R\pi_* Q_{YZ}))
\]

vanishes in the said range.

**Step 3: Proving the required vanishing in the boundary**

Let \( Y_Z = \bigcup Y_i \), where \( Y_i \) are smooth irreducible divisors. Let \( i_k : A_k \to Y_Z \) for \( k \geq 0 \) be disjoint union of the \((k+1)\)-fold intersections (hence \( A_0 = \bigcup Y_i \) and \( A_k \) is of dimension \( \dim Y - k - 1 \)). Then there is a resolution:

\[
0 \to Q_{YZ} \to i_{0*} Q_{A_0} \to i_{1*} Q_{A_1} \ldots
\]

and this yields an \( E_1 \) spectral sequence:

\[
E_1^{pq} = Gr^W_{\mathbb{H}_p}(Z, w_{\geq \text{Id}}(R\pi_* Q_{AP})) \Rightarrow Gr^W_{\mathbb{H}_{p+q}}(Z, w_{\geq \text{Id}}(R\pi_* Q_{YZ}))
\]

Now fix \( p \) and consider the triangle on \( Z \):

\[
w_{\geq \text{Id}} R\pi_* Q_{AP} \to R\pi_* Q_{AP} \to w_{\geq \text{Id}} R\pi_* Q_{AP} 
\]

giving rise to the exact long sequence:

\[
\to Gr^W_{\mathbb{H}_p}(Z, R\pi_* Q_{AP}) \to Gr^W_{\mathbb{H}_p}(Z, w_{\geq \text{Id}}(R\pi_* Q_{AP})) \to Gr^W_{\mathbb{H}_p+1}(Z, w_{\leq \text{Id}}(R\pi_* Q_{AP})) 
\]

Since \( A_p \) is smooth, the first term in (***) vanishes for \( j > 2q \) or \( j > \dim A_p = 2(\dim Y - p - 1) \) or \( j < q \) or, in case \( X \) is proper for \( j > q \). By Noetherian induction, the last term vanishes for

(i) \( j < 2(q + 1 - \dim A_p) = 2(p + q + 2 - \dim Y) \)

(ii) \( j > 2q + 2 \)
(iii) $j > 2 \dim A_p = 2(\dim Y - p - 1)$

(iv) In case $X$ is proper, we can assume that the last term vanishes for $j > q$.

To see this, note that vanishing is clear for $j > q + 1$ by the induction hypothesis.

Furthermore, $G_{r+1}^W H^{j+1}(Z, w_{\leq 1d}(R\pi_*Q_{A_p}))$ maps isomorphically onto its image under the last arrow, that is under the morphism

$$G_{r+1}^W H^{j+1}(Z, w_{\leq 1d}(R\pi_*Q_{A_p})) \to G_{r+1}^W H^{q+1}(Z, R\pi_*Q_{A_p})$$

by the induction hypothesis. In particular, the second map is 0 and we can effectively replace the last term in (***) by 0 for $j > q$.

Therefore for $0 \leq p, q$ such that $p + q = i - 1$, the middle term in (**) vanishes for

(i) $j < 2i - 2 \dim Y$:

To see this, note that since $j < 2i - 2 \dim Y = 2(p + q + 1 - \dim Y) < 2(p + q + 2 - \dim Y)$ hence the last term vanishes. If $j < q$ we saw that the first term vanishes. If $j \geq q$, we get that $q \leq j < 2(p + q + 1 - \dim Y)$ and hence $q > 2(\dim Y - q - 1) = 2 \dim A_p$, and hence it vanishes as before.

(ii) $j > 2i$:

Since $i \geq q + 1$, $j > 2i$ implies $j > 2q + 2 > 2q$ hence the first and last terms vanish.

(iii) $j > 2 \dim Y$:

Since $\dim Y > \dim Y_Z \geq \dim A_p$, $j > 2 \dim Y$ imply that the first and last terms vanish as before.

(iv) If $X$ is proper, for $j \geq i$:

Since $j \geq i > q$, the vanishings are immediate.

Hence, using the spectral sequence above we get the required vanishing of the graded pieces $G_{r+1}^W H^{j+1}(Z, w_{\leq 1d}(R\pi_*Q_{Y_Z}))$ as well, and we are done.

**Case 2: Removing the simple normal crossing condition**

In case $\pi : Y \to X$ does not satisfy that $Y_Z$ is simple normal crossing, we can always find a $\pi' : Y' \to Y$ such that the $\pi \circ \pi'$ satisfies required properties and $\dim Y = \dim Y'$, by either using resolution of singularities or de Jong’s alterations [DeJ96]. Now note that by the decomposition theorem $G_{r+1}^W H^i(X, w_{\leq 1d}R\pi_*Q_Y)$ is a summand of $G_{r+1}^W H^i(X, w_{\leq 1d}R(\pi \circ \pi')_*Q_{Y'})$, and hence vanishes when the latter does.

4.3.4. Corollary. Let $X$ be an equidimensional scheme of dimension $d_X$. Then $G_{r+1}^W EH^i(X)$ vanishes if $j \notin [0, 2i]$ (if $i \leq d_X$) or $j \notin [2(i - d_X), 2d_X]$ (if $i \geq d_X$). If $X$ is proper, it also vanishes for $j > i$ and if $X$ is smooth it vanishes for $j < i$.

Proof. For smooth $X$, $EH^i(X) = H^i(X)$ and hence its graded vanishes for $j < i$.

We can always find $Y \to X$ proper surjective with $\dim Y = \dim X$ and $Y$ smooth by either resolution of singularities or de Jong’s alterations. All the claims now follow from previous lemma (except for the case $j < 0$), noting that $EC_X$ is then a summand of $w_{\leq 1d}R\pi_*Q_Y$ (see 4.1.3).

The case $j < 0$ is immediate from 3.2.1 and 3.1.11 noting that over $\text{Spec}k$ we have $w_{D^{\geq 1d}} = w_{D^{\geq 0}}$.

4.4. Homology and cycles. Since $EH^*$ is defined using an object in $D^b(X)$ it has a natural Verdier dual homology theory of Borel-Moore type defined by

$$EH_{BM}^i(X) := H^{-i}(X, \mathbb{D}EC_X)$$
where $d = \dim X$. There is also a homology theory defined by

$$EH_i(X) := \mathbb{H}_c^{−i}(X, \mathbb{D}EC_X)$$

with a natural map $EH_∗(X) \to EH_∗^{BM}(X)$ which is an isomorphism if $X$ is proper. By [2.2.13] and (7) of [4.2.1] there are natural module actions of $EH_∗(X)$ on $EH_∗(X)$ and $EH_∗^{BM}(X)$ generalizing (and compatible with) the actions of $H^∗(X)$ on $H_∗(X)$ and $H_∗^{BM}(X)$ by cap product.

If $f : Y \to X$ is proper with image meeting the smooth locus then applying duality to the homomorphism $EC_X \to Rf_!EC_Y$ gives a natural pushforward $f_* : EH_∗^{BM}(Y) \to EH_∗^{BM}(X)$. The projection formula

$$f_∗(a \cdot f^∗(b)) = f_∗(a) \cdot b \quad (a \in EH^∗(Y), b \in EH_∗^{BM}(X))$$

follows formally from duality and the fact that $f^∗$ is a ring homomorphism.

The fundamental class homomorphism

$$Q_X \to \mathbb{D}Q_X[−2d](−d)$$

factors through a unique homomorphism

$$EC_X \to \mathbb{D}EC_X[−2d](−d).$$

(The fundamental class homomorphism factors as $Q_X \to IC_X \to \mathbb{D}Q_X[−2d](−d)$ where the two maps are dual to each other, and $Q_X \to IC_X$ factors through $EC_X$.) This can be used to define a cycle class for certain subvarieties as follows. For an irreducible $k$-dimensional subvariety $i : Z \hookrightarrow X$ meeting the smooth locus of $X$, the composition

$$Q_X \to i_*Q_Z \to i_*EC_Z \to i_*\mathbb{D}EC_Z[−2k](−k) \to \mathbb{D}EC_X[−2k](−k).$$

gives a canonical class in $EH_{2k}^{BM}(X)(−k)$ lifting the usual cycle class in $H_{2k}^{BM}(X)(−k)$. For proper morphisms $f$ with image meeting the smooth locus these cycle classes will have functoriality properties under the pushforward $f_*$.

4.4.1. Remark. In [BBFGK] the authors show that the cycle class of a subvariety $Z \subset X$ of dimension $k$ can be lifted to intersection cohomology, $IH_{2k}^{BM}(X)(−k)$, but this lift is not canonical even if $Z$ meets the smooth locus of $X$.

5. Examples

5.0.1. Example (Curves). For a curve $C$, the invariance of $EH^∗$ under normalization implies that the natural map $EH^∗(C) \to IH^∗(C)$ is an isomorphism. Thus $H^∗(C) \to EH^∗(C)$ is not an isomorphism if $C$ is singular e.g. if $C$ is the nodal cubic.

5.0.2. Example (Varieties with isolated singularities). Let $X$ be a variety with an isolated singularity $i : \{\infty\} \hookrightarrow X$. Let $\pi : Y \to X$ denote any resolution of the singularity such that $\pi^{−1}(\infty) = \cup Y_i$ is a variety with simple normal crossings. Let $j : U \to X$ denote the smooth complement. Then we have a triangle:

$$j_!Q_U \to R\pi_!Q_Y \to i_*R\pi_!Q_{\pi^{−1}(\infty)} \to$$

Since $j_!Q_U \in \mathbb{D}^{\leq 1d}$ it follows that

$$i^∗EC_X \cong i^∗w_{\leq 1d}R\pi_!Q_Y \cong w_{\leq 1d}R\pi_!Q_{\pi^{−1}(\infty)} \cong W_0H^∗(\pi^{−1}(\infty)).$$

Now since $\pi^{−1}(\infty)$ has normal crossings it follows that $W_0H^∗(\pi^{−1}(\infty))$ is the cohomology of the dual simplicial complex, i.e. the nerve of the covering by closed irreducible components. Thus $i^∗EC_X$ is the cohomology of the dual simplicial complex at infinity. (In particular this
cohomology is independent of the choice of resolution. In fact, the simple homotopy type of the dual simplicial complex is already independent of the choice, cf. \cite{Ste05,Thu07,Pay13}.

Alternatively, we can also describe $i^*EC_X$ as the weight zero part of the link, i.e. as $W_0^H(i^*Rj_*Q_U)$.

We remark that \cite{ABW11} shows that if $X$ is Cohen-Macaulay the homotopy type of the dual simplicial complex of $\pi^{-1}(\infty)$ is that of a bouquet of spheres of dimension $\dim X - 1$. For other restrictions on the stalk $i^*EC_X$ we refer to \cite{ABW11,Ste05,Pay13}.

More generally, suppose that $Z$ and $U = X - Z$ are smooth and $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ are the inclusions. If $i^*Rj_*Q_U$ is locally constant along $Z$ (e.g. if $k = \mathbb{C}$ this is true if $X = U \cup Z$ is a Whitney stratification) then the complex $(i^*EC_X)_\infty$ can be identified with the weight zero part of the cohomology of the link, i.e. $(i^*EC_X)_\infty = W_0^H((i^*Rj_*Q_U)_\infty)$.

There is also a statement in terms of fibres of a resolution of singularities if it is chosen to have simple normal crossings and such that each irreducible component maps smoothly onto $Z$.

Still more generally, if $Z \subset X$ is such that $X - Z$ is smooth then $W_0^H(Z, i^*EC_X)$ can be identified with the cohomology of the dual simplicial complex of the preimage of $Z$ in any resolution $\tau : Y \to X$ if it is of simple normal crossing type. However, in general the hypercohomology $H^*(Z, i^*EC_X)$ is not entirely of weight zero.

5.0.3. Example (Surfaces). Let $X$ be a normal surface. If the exceptional fibre in a resolution of $X$ has contractible dual graph then the previous example shows that $Q_X = EC_X$.

This is the case, for example, if $X$ is the cone over a nonsingular curve. If the curve is of genus $> 0$ then $EC_X \to IC_X$ is not an isomorphism.

On the other hand, if the components of the exceptional fibre in a resolution of singularities of $X$ are rational curves and the dual graph contains cycles then one sees that $EC_X \to IC_X$ is an isomorphism but $Q_X \to EC_X$ is not. This is the case, for example, for Hilbert modular surfaces.

5.0.4. Example (Toric varieties). For a toric variety $X$ the natural map $Q_X \to EC_X$ is an isomorphism, so that $H^*(X) = EH^*(X)$.

To see this we will use induction on the dimension of $X = X(\Delta)$ where $\Delta$ is a fan inside $N_\mathbb{R} \cong \mathbb{R}^n$ (i.e. a collection of (closed) rational polyhedral cones) the base case being easy.

By \cite{Ful93} 2.6 there is a toric resolution $\tau : Y \to X$ induced by another fan $\Delta_Y$ in $N_\mathbb{R}$.

We let $\pi$ also denote the morphism $\Delta_Y \to \Delta$ thought of as partitions of $N_\mathbb{R}$. Since this is a toric map, $\pi^{-1}\sigma = \bigcup \tau_i$ for some $\tau_i \in \Delta_Y$.

Let $j : U \hookrightarrow X$ denote the dense torus and $i : Z \hookrightarrow X$ denote the closed complement. For $Y_Z = \pi^{-1}Z$, we have a triangle:

$$j_!Q_U \to R\pi_*Q_Y \to i_*R\pi_*Q_{YZ} \to$$

Since the first term lies in $wD^{\leq 1}$, to show that the natural map $Q_X \to w_{\leq 1}R\pi_*Q_Y$ is an isomorphism, it is enough to show that the natural map $Q_Z \to w_{\leq 1}R\pi_*Q_{YZ}$ induces an isomorphism.

Let $Y_Z = \bigcup Y_i$ where $Y_i$ are irreducible. Let $W$ denote the intersection of some of the $Y_i$’s. Then $W$ is smooth and irreducible and its image $\pi(W)$ is toric, and therefore by the induction hypothesis, $w_{\leq 1}R\pi_*Q_W \cong Q_{\pi(W)}$ under the natural map.

Let $i_k : A_k \to Y_Z$ for $k \geq 0$ be the disjoint union of the $(k + 1)$-fold intersections of components of $Y_Z$. Then we have a resolution:

$$0 \to Q_{YZ} \to i_0*Q_{A_0} \to i_1*Q_{A_1} \to \cdots$$
Consider the closure $X(\sigma)$ of the torus orbit in $X$ corresponding to $\sigma \in \Delta$ and let $i_\sigma : X(\sigma) \hookrightarrow X$ be the inclusion. Let $\pi^{-1}\sigma^\circ = \cup_i \tau_i^0$. Then $\pi^{-1}X(\sigma) = \cup_i Y(\tau_i)$. Now $i_\sigma^* Q_X(\sigma') \neq 0$ iff $\sigma' \supset \sigma$, and hence if $j_\sigma : U(\sigma) \hookrightarrow X(\sigma)$ is the inclusion of the relative interior of $X(\sigma)$ then $j_\sigma^* R\pi_* Q_Y(\sigma) \neq 0$ only if $\pi \subset \pi^{-1}\sigma$, and in this case $j_\sigma^* w_{\leq 1d} R\pi_* Q_Y(\sigma) \equiv j_{\sigma'}^* Q_X(\sigma')$ for some $\sigma' \supset \sigma$.

For $x \in U(\sigma)$, applying the cohomological functor $H^0 \circ i_\sigma^* w_{\leq 1d} R\pi_*$ to the quasi-filtered object (cf. [San90, 5.2.17]) coming from the resolution of $Q_Y$ above gives a spectral sequence with

$$E_1^{p,q} = H^0(i_\sigma^* w_{\leq 1d} R\pi_* i_{pr} Q_{A_p}) \Rightarrow H^{p+q}(i_\sigma^* w_{\leq 1d} R\pi_* Q_{Z_Y}).$$

The $E_1^{p,q}$ term vanishes for $q > 0$ (by the induction hypothesis) and thus the spectral sequence is identified with the complex computing the cohomology of the cone $\sigma$ in terms of the subdivision $\pi^{-1}\sigma = \cup \tau_i$ by simplicial cones. (Since $Y$ is smooth each cone of $\Delta_Y$ is simplicial.) Since $\sigma$ is contractible we deduce that $H^*(i_\sigma^* w_{\leq 1d} R\pi_* Q_{Y_Z}) = Q[0]$.

Thus the natural map $Q_Z \to w_{\leq 1d} R\pi_* Q_{Z_Y}$ is an isomorphism on stalks and hence an isomorphism, completing the argument. (Note that the argument works equally well for varieties which are locally toric.)

### 5.0.5. Example (Atiyah flop).

The previous example also tells us that the natural map $EH^*(X) \to IH^*(X)$ may fail to be an isomorphism simply because there are toric varieties for which the natural map $H^*(X) \to IH^*(X)$ is not an isomorphism.

To consider a specific case let $X$ be the cone over a quadric, that is $X$ is given by the equation $xy - zw = 0$ in $\mathbb{A}^3$. If $Y'$ is the blowup of this cone at the origin, the exceptional fibre is isomorphic to a quadric $\mathbb{P}^1 \times \mathbb{P}^1$ in which either of the rulings can be contracted to yield two different small resolutions of $X$. Say $\pi : Y \to X$ is one of them. The resolution is small hence $IH^*(X) = H^*(Y)$, and the map $EH^*(X) \to IH^*(X)$ reduces to the map $H^*(X) \to H^*(Y)$. Using the localization sequence and noticing that the birational locus is an $\mathbb{A}^1$ fibration over the quadric, it is easy to see that this map is not an isomorphism.

Notice that in this case the two blow-downs can be used to give different ring structures on $IH^*(X)$; the natural ring structure on $EH^*(X)$ is compatible with both.

### 5.0.6. Example (Isolated rational singularities). For a complex variety $X$ with isolated rational singularities we have $EC_X = Q_X$ and hence $EH^*(X) = H^*(X)$. This follows from a result of Arapura, Bakhtary, and Włodarczyk [ABW11, Cor. 3.2] which says that if $\pi : Y \to X$ is a resolution then $W_{\emptyset} H^*(\pi^{-1}(x)) = H^0(\pi^{-1}(x)) = Q$ for any $x \in X$. The assertion then follows from Ex. 5.0.2.

Payne’s example [Pay13, §8] of an isolated singularity in which the dual complex of a resolution has the homotopy type of $\mathbb{R}P^2$ suggests that this may not be true integrally (although the integral version of $EH^*(X)$ is yet to be constructed).

### 5.0.7. Example (Locally symmetric varieties). Let $M$ be a (noncompact) locally symmetric variety, i.e. the quotient of a Hermitian symmetric domain by a neat arithmetic group of automorphisms and let $M^*$ be its Baily-Borel compactification, which is a normal complex variety. In this case there is a natural isomorphism $EH^*(M^*) = H^*(\overline{M}^{bs}, Q)$ where $\overline{M}^{bs}$ is the reductive Borel-Serre compactification. This follows from the identity

$$R_{pr} Q_{\overline{M}^{bs}} = rat(EC_{M^*})$$

where $p : \overline{M}^{bs} \to M^*$ extends the identity and $rat$ is the realization functor on mixed Hodge modules. The identity is proved in [Nai12] (or can also be deduced from [AZ12], at least
after further pushforward to a point). Let $\pi : \widetilde{M} \to M^*$ be any resolution of singularities (for example, a toroidal compactification of $M$). We then have the alternate formula

$$R\pi_* Q_{\overline{M}^{\text{rbs}}} = \text{rat}(w_{\leq 14} R\pi_* Q_{\overline{M}})$$

Thus the pullback $\pi^* : H^*(M^*) \to H^*(\widetilde{M})$ factors through $H^*(\overline{M}^{\text{rbs}})$. This is a result of [GT99] for toroidal compactifications.

The singularities of $M^*$ are log canonical ([Ko97, 3.7.1]), and hence du Bois. In general, $H^*(M^*) \to H^*(\overline{M}^{\text{rbs}})$ is not an isomorphism (not even if the singularities are isolated), so that $EH^*(M^*)$ differs from the ordinary cohomology of $M^*$ in this case.

5.0.8. Example (Hilbert-Blumenthal varieties). An interesting locally symmetric example is that of a Hilbert-Blumenthal variety $M$ and its Baily-Borel compactification $M^*$, which has isolated singularities at $M^* - M$. In this case one can show using the explicit computation of the local intersection cohomology that $EC_{M^*} \to IC_{M^*}$ is an isomorphism. (Since the singularity is isolated this amounts to the statement that the local cohomology of $IC_{M^*}$ is entirely of weight zero.) It follows that $EH^*(M^*) = IH^*(M^*)$ so that $IH^*(M^*)$ is a ring.

5.0.9. Example (Picard modular varieties). If $M$ is a Picard modular variety and $M^*$ its Baily-Borel compactification as above, then there is a canonical smooth resolution which is minimal. However, since $\overline{M}^{\text{rbs}} = M^*$ in this case, $EC_{M^*} = Q_{M^*}$ as one sees from the discussion above; so $EH^*(M^*) = H^*(M^*)$.

5.0.10. Example (Siegel modular threefolds). We give an example to show that the stalk of $EC_X$ at $x \in X$ is not simply the weight zero piece of the fibre over $x$ in a resolution.

We use a locally symmetric example as the availability of the space $\overline{M}^{\text{rbs}}$ simplifies the discussion. The boundary of the Baily-Borel compactification of a Siegel modular threefold $M$ is naturally stratified as the union of (open) modular curves and cusps. For example, $M^*$ is a morphism of locally symmetric varieties coming from a totally geodesic embedding of Hermitian symmetric domains. Thus $f : M_H \to M$ is a morphism of locally symmetric varieties coming from a totally geodesic embedding of Hermitian symmetric domains. Thus $f : M_H \to M$ is finite onto a closed subvariety. Then $f$ extends to $\widetilde{M} \to M^{\text{rbs}}$ and the image of $f$ meets the smooth locus $M$. In [Nai10] we study the cycle class $cl(f(M_H)) \in EH^{BM}_*(M^*) = EH_*(M^*)$ and use it to prove injectivity of pullbacks $EH^*(M^*) \to EH^*(M_H^*)$ in some cases.

These results depend in an essential way on the de Rham model available for $EH^*(M^*) = H^*(\overline{M}^{\text{rbs}})$, which allow us to use arguments about automorphic forms.

References


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