BERTINI'S THEOREM ABOUT NONSINGULARITY

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A . In this lecture, we prove Bertini's theorem, which says that a generic hyperplane section of a smooth projective variety over an algebraically closed field k is smooth. As applications, we shall discuss the existence of nonsingular hypersurfaces of degree d, for any $d \geq 1$ and that of complete intersections of height r, for any $1 \leq r < n$, in $\mathbb{P}^n_k$.

1. I

One of the most beautiful and powerful aspects of algebraic geometry is the tendency of a family of algebraic varieties to be often parameterized, in a natural way, by another algebraic variety.[1] One class of results reflecting this idea is the Bertini theorems, which say that under appropriate conditions, a *generic* hyperplane section of a projective variety has a certain property of the ambient variety. The aim of this article is to prove the following version: if $X \subseteq \mathbb{P}^n_k$ is a nonsingular projective variety, then a generic hyperplane in $\mathbb{P}^n_k$ intersects X in a nonsingular variety; that means, the set of such hyperplanes is a Zariski-open dense subset of the parameterizing variety, which itself is an n-dimensional projective space, as we shall see in due course.

Throughout the article, k will denote an algebraically closed field, unless mentioned otherwise. We need to first define the notion of nonsingularity of a variety at a point. We give the intrinsic definition of nonsingularity given by Zariski in terms of local rings at points.

Recall that for a noetherian local ring A with unique maximal ideal $\mathfrak{m}$, $\mathfrak{m}/\mathfrak{m}^2$ is an $A/\mathfrak{m}$-vector space and we have $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.

Definition 1.1. Let A be a noetherian local ring with unique maximal ideal $\mathfrak{m}$. A is said to be a *regular local ring* if $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Definition 1.2. A variety X is said to be *nonsingular* at a point $p \in X$ if the local ring $\mathscr{O}_{p,X}$ at p is a regular local ring. X is called a *nonsingular variety* if it is nonsingular at every point $p \in X$.

For affine varieties, nonsingularity could be defined using the Jacobian criterion: an affine variety $X \subseteq \mathbb{A}^n_k$ of dimension d is said to be nonsingular at $p \in X$ if the Jacobian matrix at p, $\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j}$, has rank $n - d$, where $X = V(f_1, \ldots, f_r)$. It can be proved that this definition is independent of the choice of generators of the ideal of X, but the definition apparently depends on the embedding of X in the affine space. This definition is equivalent to the above definition due to Zariski, which is more general as it readily generalizes to

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[1] An algebraic variety is the common zero set of a set of polynomials; note that we do not assume irreducibility.

arbitrary varieties and schemes. One can also prove the Jacobian criterion for projective varieties: a projective variety $Y = V(g_1, \dots, g_r)$ is nonsingular at a point p if and only if the Jacobian matrix at p, $\left(\frac{\partial g_i}{\partial x_j}(p)\right)_{i,j}$, has rank $n - d$, where $\dim Y = d$. The Jacobian criterion is very useful in giving the defining equations of an affine or projective variety.

We shall require the following results, which we will not prove here to avoid a lengthy interruption in the exposition.

Theorem 1.3 (Main theorem of elimination theory)**.** *The second projection* $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$ *is a closed map, that is, it takes Zariski-closed sets in* $\mathbb{P}^n \times \mathbb{P}^m$ *to Zariski-closed sets in* $\mathbb{P}^m$*.*

Proof. See Mumford [3, §2C, Theorem 2.23, page 33]. □

Theorem 1.4 (Dimension of fibres)**.** *Let* X *and* Y *be projective varieties of dimension* n *and* m *respectively, and* $f : X \to Y$ *be a surjective morphism. Then* $m \leq n$*, and*

(1) *For any* $y \in Y$*, every irreducible component of the fibre* $f^{-1}(y)$ *has dimension* $\geq n - m$*.*

(2) *There exists a nonempty open set* $U \subseteq Y$ *such that* $\dim f^{-1}(y) = n - m$*, for all* $y \in U$*.*

Proof. See Shafarevich [4, Chapter I, 6.3, Theorem 7, page 76]. □

2. T M T

We are now ready to prove the main theorem of the article. Any hyperplane in $\mathbb{P}^n_k$ is given by the zero-set of a homogeneous linear polynomial $\sum_{i=0}^n a_i x_i$, where x_i's are the coordinates on $\mathbb{P}^n_k$, $a_i \in k$, and not all the a_i are zero. Thus, every hyperplane uniquely determines a point (in this case, $[a_0 : \cdots : a_n]$) in a projective space of dimension n over k, called the *dual projective space* to $\mathbb{P}^n_k$; it is denoted by $(\mathbb{P}^n_k)^*$.

Theorem 2.1 (Bertini)**.** *Let* X *be an irreducible, nonsingular closed subvariety of* $\mathbb{P}^n_k$*, where* k *is an algebraically closed field. Then there exists a hyperplane* $H \subseteq \mathbb{P}^n_k$*, not containing* X*, such that* $H \cap X$ *is a nonsingular variety (of dimension* $\dim X - 1$*). Furthermore, the set of hyperplanes with this property forms a dense open subset of the dual projective space* $(\mathbb{P}^n_k)^*$ *of hyperplanes in* $\mathbb{P}^n_k$*.*

Proof. Let $\dim X = d$. Let $p \in X$ be a closed point. Set

$$B_p := \{H \in (\mathbb{P}^n_k)^* \mid X \not\subseteq H \text{ and } p \text{ is not a regular point of } H \cap X \textit{ or } X \subseteq H\},$$

the set of "bad" hyperplanes at x. Consider $B \subseteq X \times (\mathbb{P}^n_k)^*$ defined as follows:

$$B := \{(x, H) \mid p \in X \text{ is a closed point and } H \in B_p\}.$$

Note that B is a projective variety; the defining equations of B are as follows: Suppose that $X = V(f_1, \dots, f_r)$. Define $f_{r+1}(x) := a_0x_0 + \cdots + a_nx_n$. A point $p \in X$ lies on $H \cap X$ if and only if p satisfies each of the equations $f_1, \dots, f_r$ and f_{r+1}. p is not a regular point of $H \cap X$ if and only if the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1 \leq i \leq r+1,\ 0 \leq j \leq n}$ has rank $< n - \dim H \cap X = n - d + 1$, that is, if and only if determinants of all the $(n - d + 1) \times (n - d + 1)$ minors of the Jacobian matrix vanish. All these polynomials together constitute the defining equations for B. We claim that The first projection $\pi_1 : B \to X$ is surjective, and that $\dim B = n - 1$. In order to prove these, it suffices to prove that for any closed point $p \in X$, the fibre $\pi_1^{-1}(p)$ is a projective space of dimension $n - d - 1$; for, then from Theorem 1.4, it follows that $\dim B = n - 1$.

We first characterize the “bad” hyperplanes at p. Let $x_0, \ldots, x_n$ be the coordinates on $\mathbb{P}^n_k$ and $a_0, \ldots, a_n$ be the coordinates on the dual projective space $(\mathbb{P}^n_k)^*$. Let V denote the k-vector space of all the homogeneous linear polynomials in $x_0, \ldots, x_n$. Now, there exists a hyperplane which does not pass through $p = (p_0, \ldots, p_n)$ (for, otherwise, we would have $p = (0, \ldots, 0)$, which is not possible); so without loss of generality, we may assume that the hyperplane H_0 defined by $x_0 = 0$ does not pass through p. Define a k- linear map $\varphi_p : V \to \mathscr{O}_{p,X}/\mathfrak{m}_p^2$ by

$$\varphi_p : \sum_{i=0}^{n} a_i x_i \mapsto \sum_{i=0}^{n} a_i \frac{x_i}{x_0}.$$

Since k is algebraically closed, the maximal ideal m_p of $\mathscr{O}_{p,X}$ is generated by linear polynomials in $x_1/x_0, \ldots, x_n/x_0$. Hence, it follows that φ_p is surjective. Since X is nonsingular, $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim X = d$, whence $\dim_k \mathscr{O}_{p,X}/\mathfrak{m}_p^2 = d + 1$. Since $\dim_k V = n + 1$, we have $\dim_k \ker \varphi_p = n - d$. Note that for $f \in V$, $\varphi_p(f) = 0$ implies that f vanishes in a neighbourhood of p in X, and hence, on the whole of X; thus, $X \subseteq H$, where H is the hyperplane defined by $f = 0$. Next, observe that $p \in H \cap X$ if and only if $\varphi_p(f) \in \mathfrak{m}_p$. In this case, the ring $\mathscr{O}_{p,X}/(\varphi_p(f))$ is regular if and only if $\varphi_p(f) \in \mathfrak{m}_p - \mathfrak{m}_p^2$. Therefore, we conclude that the hyperplanes in B_p are exactly the elements of the projectivization of the k-vector space $\ker \varphi_p$ (for, scalar multiples of an element $f \in V$ define the same hyperplane), whence $\dim B_p = n - d - 1$. The claim now follows by observing that $\pi_1^{-1}(p)$ is isomorphic to B_p.

Having proved the claim, we see from Theorem 1.3 that the image of B under the second projection $\pi_2 : B \to \mathbb{P}^n_k$ is a proper closed subset of $(\mathbb{P}^n_k)^*$. Let $U = (\mathbb{P}^n_k)^* - \pi_2(B)$. Then it is easy to see that U is a required dense open set in the theorem.

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3. A B ’ T

Existence of Nonsingular Curves. We now apply Bertini’s theorem to prove the existence of nonsingular hypersurfaces of degree d, for any $d \geq 1$. We shall first introduce a very useful and improtant construction called the *Veronese* or *d-uple embedding*, which embeds $\mathbb{P}^n_k$ into $\mathbb{P}^N_k$ as a projective subvariety, where $N = \binom{n+d}{d} - 1$. This morphism $\rho_d : \mathbb{P}^n_k \to \mathbb{P}^N_k$ is given by

$$\rho_d : x = [x_0 : \cdots : x_n] \mapsto [M_0(x) : \cdots : M_N(x)],$$

where $M_0, \ldots, M_N$ are all the monomials of degree d in $x_0, \ldots, x_n$. The corresponding homomorphism of coordinate rings $\psi_d : k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ is given by sending y_i to $M_i(x)$. Now, let $n \geq 2$. Note that the image of a nonsingular hypersurface of degree d under the ρ_d is a hyperplane in $\mathbb{P}^N_k$. Taking $X = \mathbb{P}^n_k$ inside $\mathbb{P}^N_k$ and applying Bertini’s theorem (Theorem 2.1), we see that for a generic hyperplane H in $\mathbb{P}^N_k$, $H \cap X$ is nonsingular. Therefore, for any $d \geq 1$, there exist nonsingular hypersurfaces of degree d in $\mathbb{P}^n_k$; in fact, they form a dense open set in $(\mathbb{P}^N_k)^*$.

Complete Intersections in $\mathbb{P}^n_k$. A closed subscheme Y of $\mathbb{P}^n_k$ is said to be a complete intersection if the homogeneous ideal $I(Y)$ of Y in $S = k[x_0, \ldots, x_n]$ is generated by $r = n - \dim Y$ elements. (A closed subscheme of $\mathbb{P}^n_k$ is the Proj of S/I, for some homogeneous ideal

$I \subseteq S$.) Suppose that we are given integes $d_1, \dots, d_r \geq 1$, with $r < n$. Using Bertini's theorem and the argument used in the above subsection repeatedly, we see that there exist nonsingular hypersurfaces $H_1, \dots, H_r$ in $\mathbb{P}^n_k$, $H_i = V(f_i)$ with $\deg f_i = d_i$ such that the scheme $Y = H_1 \cap \dots \cap H_r$ is irreducible and nonsingular of codimension r in $\mathbb{P}^n_k$. We shall now prove the following: A closed subscheme of codimension r in $\mathbb{P}^n_k$ is a complete intersection if and only if there exist hypersurfaces $H_1, \dots, H_r$ such that $Y = H_1 \cap \dots \cap H_r$ as schemes. The "only if" part is easy to see. Before proving the "if" part, we require some concepts from commutative algebra. [2]

Given any ideal in a noetherian ring A, a prime ideal $\mathfrak{p}$ of A is said to be an *associated prime* of I if $\mathfrak{p}$ is the annihilator of some element of A/I. There are only finitely many such primes associated to a given ideal I. The prime ideals of A minimal over I occur among the primes associated to I. An ideal $Q \subseteq \mathfrak{p}$ is said to be $\mathfrak{p}$-*primary* if $\sqrt{Q} = \mathfrak{p}$ and if for any $a, b \in A$ with $ab \in Q$ and $a \notin Q$, we have $b^n \in Q$ for some n. Any ideal I of A can be expressed as the intersection of primary ideals. Since the intersection of ideals primary to a given prime $\mathfrak{p}$ is again $\mathfrak{p}$-primary, I can be expressed as an intersection of ideals that are primary to distinct primes. If this done so that none of the associated primes of I are left out, the expression is called a *primary decomposition* of I and the primary ideals involved are called *primary components of* I. The associated primes of I are thus the radicals of the primary components. The primary component of I corresponding to a given prime $\mathfrak{p}$ is uniquely determined by I if $\mathfrak{p}$ is minimal over I. Such primary components are called *isolated components* and the rest are called *embedded components*.

Let A be a commutative local ring with maximal ideal $\mathfrak{m}$ and M be an A-module. Let $a_1, \dots, a_n$ be a sequence of elements in $\mathfrak{m}$. We say that $a_1, \dots, a_n$ is an *M-regular sequence* in $\mathfrak{m}$ if for each $1 \leq i \leq n$, a_i is not a zerodivisor in $M/(a_1, \dots, a_{i-1})M$, and $M \neq \sum a_i M$. If, in addition, A is noetherian, and M is a finitely generated A-module, every M-regular sequence in $\mathfrak{m}$ can be extended to a *maximal* M-regular sequence in $\mathfrak{m}$. In this setting, any two maximal M-regular sequence in $\mathfrak{m}$ have the same length. The length of the maximal M-regular sequence in $\mathfrak{m}$ under these assumptions is called the *depth* of M and is denoted by $\operatorname{depth}_A(M)$. If $M = A$, we just write $\operatorname{depth} A$ for the depth of A. If $M \neq 0$, it can be proved that $\operatorname{depth} A \leq \dim A$.

Definition 3.1. Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. A is said to be a *Cohen-Macaulay local ring* if $\operatorname{depth} A = \dim A$, or if $A = 0$. A noetherian ring A is said to be a *Cohen-Macaulay ring* if $A_{\mathfrak{p}}$ is a Cohen-Macaulay local ring, for every prime ideal $\mathfrak{p}$ of A. This is equivalent to saying that $A_{\mathfrak{m}}$ is a Cohen-Macaulay local ring, for every maximal ideal $\mathfrak{m}$ of A.

Let A be a noetherian ring and I an ideal of A; let $\operatorname{Ass}_A(A/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_l\}$. We say that I is *unmixed* if $\operatorname{ht}(\mathfrak{p}_i) = \operatorname{ht}(I)$, for all i. Note that I is unmixed if and only if A/I has no embedded primes. We say that the *unmixedness theorem holds in A* if the following is true: every ideal of height r, which is generated for r elements, is unmixed. It is not hard to see that the unmixedness theorem holds in A if it holds in $A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of A.

Theorem 3.2. *Let A be a noetherian ring. Then A is Cohen-Macaulay if and only if the unmixedness theorem holds in A.*

Proof. See Matsumura [2, Chapter 6, Theorem 32, page 110]. □

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[2] We only give an outline of results in this section and refer the reader to Matsumura [2] for a detailed exposition.

Theorem 3.3. *Let A be a Cohen-Macaulay ring. Then the polynomial ring $A[x_1, \ldots, x_n]$ is also Cohen-Macaulay.*

Proof. See Matsumura [2, Chapter 6, Theorem 33, page 111]. $\square$

We are now set to prove that if there exist hypersurfaces $H_1, \ldots, H_r$ with $H_i = V(f_i)$ such that $Y = H_1 \cap \ldots \cap H_r$ as schemes, then Y is a complete intersection. Using Theorems 3.2 and 3.3, we conclude that the unmixedness theorem holds in $S = k[x_0, \ldots, x_n]$. Since $Y = H_1 \cap \ldots \cap H_r$ as schemes, their ideal sheaves are the same. Therefore, if $(f_1, \ldots, f_r) \neq I(Y)$, then we must have $(f_1, \ldots, f_r) = I(Y) \cap J$, where J is an ideal primary to the irrelevant maximal ideal $(x_0, \ldots, x_n)$. This is because $V((x_0, \ldots, x_n)) = \emptyset$. Now since $H_1 \cap \ldots \cap H_r$ has codimension r, we have $\mathrm{ht}(f_1, \ldots, f_r) = r$, whence it must be unmixed; that is , it has no primary components of codimension $> r$. But $\mathrm{ht}(J) = \mathrm{ht}(x_0, \ldots, x_n) = n + 1 > r$, a contradiction. Therefore, we must have $I(Y) = (f_1, \ldots, f_r)$, so Y is a complete intersection of codimension r.

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[1] Hartshorne, R., *Algebraic Geometry*.
[2] Matsumura, H., *Commutative Algebra*.
[3] Mumford, D., *Algebraic Geometry I: Complex Projective Varieties*.
[4] Shafarevich, I., *Basic Algebraic Geometry I*.

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