THE LEVEL OF FIELDS, RINGS AND TOPOLOGICAL SPACES

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A . An important invariant of a (nonreal) field is its level, which is defined to be the least positive integer n such that -1 is a sum of n squares in that field. Using Pfister's theory of multiplicative quadratic forms, we prove that the level of a field must be a power of 2 and present a construction of fields of any prescibed level 2^n. The definition of level applies for any nonzero ring with unity and it is proved that given any positive integer n, there exists a ring with level n. This is deeply connected to a level-theory for topological spaces with involution started by Dai and Lam.

1. I

In a fundamental paper of Artin and Schreier [1], it was shown that the algebraic investigation of fields requires us to distinguish between formally real (that is, fields in which -1 cannot be written as a sum of squares) and nonreal (that is, fields in which -1 is a sum of squares) fields. Artin and Schreier characterized formally real fields to be the fields possessing at least one linear ordering. In order to study nonreal fields, one has to study invariants of them. A most natural invariant of a nonreal field is its *level*, the least n such that -1 is a sum of n squares in that field. The first interesting result is the fact that the level of a field is a power of 2, which was proved by Pfister in 1965 using a special kind of multiplicative quadratic forms, after it had remained an unsolved problem for more than thirty years. Pfister also constructed fields of a given finite level 2^n, using certain results in function fields by himself and Cassels. Pfister's fundamental discovery revolutionized the theory of quadratic forms and became a powerful tool in the research in this area.

The definition of level can also be carried to rings. The great surprise however was the discovery of commutative rings, in fact, integral domains, with prescribed finite level by Dai, Lam and Peng in 1980. This is deeply connected to a level-theory for topological spaces with involution started by Dai and Lam in 1984 by further developing the considerations in the proof of their result about the level of rings. They proved that the topological level of a space X with an involution equals the algebraic level of the $\mathbb{R}$-algebra of $\mathbb{C}$-valued functions on X respecting involutions. This article is aimed to give an account of these results regarding the level of fields, rings and topological spaces.

2. P

Generalities on Quadratic Forms. [1]

Throughout the article, a field will mean a field of characteristic $\neq 2$, unless mentioned otherwise.

[1] We only give an outline of results and refer the reader to [5] for a detailed exposition.

Definition 2.1. *An (n-ary) quadratic form over a field F is a homogeneous polynomial of degree 2 in n variables over F.*

A quadratic form over F has the general form

$$q(x_1,\ldots,x_n) = \sum_{1\,\le\, i,j\,\le\, n} a_{ij}x_ix_j \in F[x_1,\ldots,x_n].$$

Since we have $\operatorname{char} F \neq 2$, we can replace the coefficients a_{ij} by $\frac{a_{ij}+a_{ji}}{2}$ without changing the form. In this way, q uniquely determines a symmetric matrix $M_q = (a_{ij})$ and in matrix notations, we have

$$q(x_1,\ldots,x_n) = x^t M_q x,$$

where x stands for $(x_1,\ldots,x_n)$, viewed as a column vector.

Definition 2.2. *Two n-ary quadratic forms q and q' are said to be equivalent if there exists a nonsingular linear transformation $A \in GL_n(F)$ such that $q'(x) = q(Ax)$ for all $x \in F^n$; in such a case, we write $q \cong q'$.*

Equivalence of two quadratic forms is clearly an equivalence relation. Note that two quadratic forms q and q' are equivalent if and only if $M_{q'} = A^tM_qA$, for some $A \in GL_n(F)$.

Every quadratic form q determines a symmetric matrix M_q, which in turn, determines a symmetric bilinear form B_q on the F-vector space F^n. In terms of q, B_q can be given by

$$B_q(x,y) = \frac{1}{2}[q(x+y) - q(x) - q(y)].$$

In case M_q is a diagonal matrix with diagonal $(a_1,\ldots,a_n)$, we abbreviate the form q by $\langle a_1,\ldots,a_n\rangle$.

Definition 2.3. *Let q be an n-ary quadratic form over a field F. Then the determinant of q is defined to be the determinant of the matrix M_q and is denoted by $\det q$. A quadratic form is said to be regular if $\det q \neq 0$.*

Note that if two n-ary quadratic forms q and q' are equivalent, then $M_{q'} = A^tM_qA$ for some nonsingular matrix A. Hence, $\det q' = \det M_{q'} = \det M_q \cdot (\det A)^2 = \det q \cdot (\det A)^2$. This shows that an equivalence class $[q]$ of quadratic forms uniquely determines an element of F/F^2. If q is regular, then $[q]$ determines a unique element of F^*/F^{*2}.

There are two basic operations on quadratic forms. If q is an n-ary quadratic form and q' an m-ary quadratic form, then we define their *orthogonal direct sum* $q \oplus q'$ to be $(n+m)$-ary quadratic form associated with the symmetric matrix

$$M_{q\oplus q'} := \begin{pmatrix} M_q & 0 \\ 0 & M_{q'} \end{pmatrix}.$$

The *tensor product* of q and q' is defined to be the the (nm)-ary quadratic form $q \otimes q'$ associated with the symmetric matrix

$$M_{q\otimes q'} := \begin{pmatrix} a_{11}M_{q'} & a_{12}M_{q'} & \cdots & a_{1n}M_{q'} \\ a_{21}M_{q'} & a_{22}M_{q'} & \cdots & a_{2n}M_{q'} \\ \vdots & \vdots & \ddots & M_{q'} \\ a_{n1}M_{q'} & a_{n2}M_{q'} & \cdots & a_{nn}M_{q'} \end{pmatrix}, \text{ where } M_q = (a_{ij}).$$

Thus, $M_{q\otimes q'}$ is just the *Kronecker product* of matrices M_q and $M_{q'}$. In particular, we have $\langle a\rangle \otimes \langle b\rangle \cong \langle ab\rangle$ and $\langle 1,a\rangle \otimes \langle 1,b\rangle \cong \langle 1,a,b,ab\rangle$ for all $a,b \in F$.

Definition 2.4. *An n-ary quadratic form q over F is said to be isotropic if there exists a nonzero element $x \in F^n$ such that $q(x) = 0$. q is called anisotropic if it is not isotropic. A quadratic form of type $\langle 1, -1\rangle \oplus \cdots \oplus \langle 1, -1\rangle$ is called hyperbolic.*

It is not difficult to verify that the operations $\oplus$ and $\otimes$ are preserved under equivalence of quadratic forms. This makes the set of equivalence classes of quadratic forms into a commutative semiring. The *Grothendieck completion* of this semiring, obtained by including additive inverses of elements to the semiring *formally* is a ring, called the *Grothendieck-Witt ring* of F and denoted by $\widehat{W}(F)$. The quotient ring of $\widehat{W}(F)$ by the hyperbolic forms is a ring with the form $\langle 1, -1\rangle$ serving as the zero and the form $\langle 1\rangle$ as the unity, called the *Witt ring* of F.

Theorem 2.5. *Let F be a field and q be an n-ary quadratic form over F. Then q is equivalent to a diagonal form $\phi(x) = \sum_{i=1}^{n} a_i x_i^2$, for some $a_1, \ldots, a_n \in F$.*

Proof. We use induction on n. If $q(x) = 0$ for all x, then $M_q = 0$ with respect to any basis. Assume $q(x_1) = a_1 \neq 0$ for some $x_1 \in F^n$ and consider the subspace

$$V = (Fx_1)^{\perp} = \{y \in F^n \mid B_q(y, x_1) = 0\}.$$

$q(x_1) \neq 0$ implies that $\dim V = n - 1$, so we have $F^n = (Fx_1) \oplus V$ and $q = q_1 \oplus q_2$ with $q_1 = q\,|_{Fx_1} = \langle a_1\rangle$ and $q_2 = q\,|_V$. By induction, $q_2 \cong \langle a_2, \ldots, a_n\rangle$ for some $a_2, \ldots, a_n \in F$, and the proof follows.

□

Note. In the case $q \neq 0$ in the above proof, we can take for a_1 any nonzero element of F represented by q.

Corollary 2.6. *Let F be a field and $a, b \in F^*$. If the quadratic form $\langle a, b\rangle$ over F represents $c \in F^*$, then*

$$\langle a, b\rangle \cong \langle c, abc\rangle.$$

Proof. By the above note, we have $\langle a, b\rangle \cong \langle c, d\rangle$ for some $d \in F$. Comparing determinants, we conclude that $abcd \in F^{*2}$, whence $\langle d\rangle \cong \langle abc\rangle$, finishing the proof. □

Pfister Forms. Let F be a field of characteristic $\neq 2$. Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be algebraically independent indetermiantes over F. We let x and y stand for $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively. We shall denote the rational function fields $F(x_1, \ldots, x_n)$ and $F(x_1, \ldots, x_n, y_1, \ldots, y_n)$ by $F(x)$ and $F(x, y)$ respectively.

Definition 2.7. *Let φ be an n-ary quadratic form over a field F. φ is said to be multiplicative over F if there exist $z_1, \ldots, z_n \in F(x, y)$ such that*

$$\varphi(x)\varphi(y) = \varphi(z),$$

where $z = (z_1, \ldots, z_n)$. φ is said to be strictly multiplicative over F if z can be chosen to depend linearly on y, that is, if there exists a matrix $T_x \in GL_n(F(x))$ such that

$$\varphi(x)\varphi(y) = \varphi(T_x y).$$

In other words, φ is strictly multiplicative if and only if $\varphi(x)\varphi \cong \varphi$ over $F(x)$. Further, observe that the nonzero elements in F represented by a multiplicative quadratic form constitute a group under multiplication.

Pfister considered the form $\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$, given n elements $a_1, \ldots, a_n \in F^*$. He proved in [6] that this form is multiplicative and used it to prove that the level of any field is a power of 2. Such a form is now called an *n-fold Pfister form*, in Pfister's honour.

Theorem 2.8 (Pfister, Witt)**.** *Let F be a field and $a_1, \ldots, a_n \in F^*$. Then the n-fold Pfister form $\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ is strictly multiplicative.*

Proof. We prove this by induction on n. If $n = 1$, then the theorem clearly holds as can be seen from the identity

$$(s^2 + at^2)(x^2 + ay^2) = (sx - aty)^2 + a(sy + tx)^2$$

Assume $n > 1$ and write $\varphi = \psi \otimes \langle 1, a\rangle$, where $a = a_n$. Then $\varphi = \psi \oplus a\psi$. By induction ψ is strictly multiplicative and hence, we have $\varphi(x)\varphi \cong \varphi$ over $F(x)$. Using this, we conclude that

$$\begin{aligned}\varphi &= \psi \oplus a\psi \\ &\cong \psi(x)\psi \oplus a\psi(y)\psi \\ &\cong \psi \otimes \langle \psi(x), a\psi(y)\rangle,\end{aligned}$$

where the equivalence is over $F(x, y)$. Now, observe that $\varphi(x, y) = \psi(x) + a\psi(y)$ is nonzero in $F(x, y)$ and is represented by the form $\langle \psi(x), a\psi(y)\rangle$. Therefore, by Corollary 2.6, we have

$$\langle \psi(x), a\psi(y)\rangle \cong \langle \varphi(x, y), \varphi(x, y)\psi(x)a\psi(y)\rangle \cong \varphi(x, y)\langle 1, a\psi(x)\psi(y)\rangle,$$

whence it follows that

$$\begin{aligned}\varphi &\cong \psi \otimes \langle \psi(x), a\psi(y)\rangle \\ &\cong \psi \otimes \varphi(x, y)\langle 1, a\psi(x)\psi(y)\rangle \\ &\cong \varphi(x, y)[\psi \otimes \langle 1, a\psi(x)\psi(y)\rangle] \\ &\cong \varphi(x, y)[\psi \oplus a\psi(x)\psi(y)\psi] \\ &\cong \varphi(x, y)[\psi \oplus a\psi] \\ &\cong \varphi(x, y)\varphi,\end{aligned}$$

over $F(x, y)$. Therefore, φ is strictly multiplicative, and the theorem is proved. □

3. T L F

A field is said to be *formally real* if -1 is not a (finite) sum of squares in that field; otherwise, it is called *nonreal*.

Definition 3.1. *The level of a nonreal field F is defined to be the least positive integer n such that -1 is a sum of n squares in F:*

$$s(F) := \{n \in \mathbb{N} \mid -1 = x_1^2 + \ldots + x_n^2 \text{ for some } x_i \in F\}.$$

The level of a formally real field is defined to be ∞.

The letter s stems from the German word *stufe* for level.

Examples 3.2. (1) $s(\mathbb{R}) = \infty$; and $s(\mathbb{C}) = 1$ since $i^2 = -1$ in $\mathbb{C}$.

(2) If F is field of characteristic 2, then $-1 = 1$ in F, so $s(F) = 1$.
(3) If $\mathbb{F}_p$ denotes the finite field with p elements, then

$$s(\mathbb{F}_p) = \begin{cases} 1 & \text{if } p \equiv 1(\text{mod } 4), \\ 2 & \text{if } p \equiv 3(\text{mod } 4). \end{cases}$$

Theorem 3.3 (Pfister)**.** *Let F be a field. If $s(F) < \infty$, then $s(F) = 2^n$ for some non-negative integer n.*

Proof. Assume $s(F) = n$ and let 2^m be the highest power of 2 less than or equal to n. Let φ denote the Pfister form

$$\underbrace{\langle 1, 1\rangle \otimes \cdots \otimes \langle 1, 1\rangle}_{\text{m}}.$$

By hypothesis, there exist elements $x_1, \ldots, x_n \in F$ such that $-1 = x_1^2 + \cdots + x_n^2$. Suppose, if possible, that $2^m < n$. Then

$$x_1^2 + \cdots + x_{2^m}^2 = -1(1 + x_{2^m+1}^2 + \cdots + x_n^2).$$

Put $d = 1 + x_{2^m+1}^2 + \cdots + x_n^2$. Since $s(F) = n$, d cannot be 0, so we have

$$-1 = \frac{1}{d}(x_1^2 + \cdots + x_{2^m}^2) = d\left[\left(\frac{x_1}{d}\right)^2 + \cdots + \left(\frac{x_n}{d}\right)^2\right].$$

d, being a sum of not more that 2^m squares is clearly represented by φ and so is $\left(\frac{x_1}{d}\right)^2 + \cdots + \left(\frac{x_{2^m}}{d}\right)^2$. Since φ is multiplicative (Theorem 2.8), we have

$$d\left[\left(\frac{x_1}{d}\right)^2 + \cdots + \left(\frac{x_{2^m}}{d}\right)^2\right] = y_1^2 + \cdots + y_{2^m}^2 = -1,$$

for some $y_i \in F$, a contradiction to our assumption that $s(F) = n$. Hence, we must have $s(F) = n = 2^m$. □

4. F P L

Lemma 4.1. *Let F be a field, $a_1, \ldots, a_{2^n} \in F$ and put $a = a_1^2 + \cdots + a_{2^n}^2$. Then there exists a $2^n \times 2^n$ matrix A over F with first row $(a_1, \ldots, a_{2^n})$ such that $AA^t = A^tA = aI_{2^m}$, where I_{2^m} denotes the $2^n \times 2^n$ identitiy matrix.*

Proof. We shall prove this by induction on n. The result trivially holds for $n = 0$. Assume $n > 0$. Write $a = b + c$, where $b = a_1^2 + \cdots + a_{2^{n-1}}^2$ and $c = a_{2^{n-1}+1}^2 + \cdots + a_{2^n}^2$. By induction, there exist $2^{n-1} \times 2^{n-1}$ matrices B and C over F such that the first row of B is $(a_1, \ldots, a_{2^{n-1}})$, the first row of C is $(a_{2^{n-1}+1}, \ldots, a_{2^n})$ and

$$BB^t = B^tB = bI_{2^{n-1}},$$

$$CC^t = C^tC = cI_{2^{n-1}}.$$

If $b \neq 0$, an easy computation shows that the matrix

$$A = \begin{pmatrix} B & C \\ -b^{-1}B^tC^tB & B^t \end{pmatrix},$$

solves the problem. If $b = 0$ but $c \neq 0$, then we set

$$A = \begin{pmatrix} B & C \\ C^t & -c^{-1}C^tB^tC \end{pmatrix},$$

and if $b = c = 0$, then the matrix

$$A = \begin{pmatrix} B & C \\ B & C \end{pmatrix},$$

has the desired properties.

□

In particular, the above lemma tells us that any unimodular row $(a_1, \dots, a_{2^n})$ over F (that is, $a_1^2 + \cdots + a_{2^n}^2 = 1$) can be completed to give an orthogonal matrix over F.

Lemma 4.2. *Let F be a field, $m = 2^n$, and $a_1, \dots, a_m, b_1, \dots, b_m \in F$. Then there exist $c_2, \dots, c_m \in F$ such that*

$$(a_1^2 + \cdots + a_m^2)(b_1^2 + \cdots + b_m^2) = \left(\sum_{i=1}^{m} a_ib_i\right)^2 + c_2^2 + \cdots + c_m^2.$$

Proof. Put $a = a_1^2 + \cdots + a_m^2$ and $b = b_1^2 + \cdots + b_m^2$. By Lemma 4.1, there exist $m \times m$ matrices A and B over F with first rows $(a_1, \dots, a_m)$ and $(b_1, \dots, b_m)$ respectively such that

$$AA^t = A^tA = aI_m$$

and

$$BB^t = B^tB = bI_m.$$

Let $C = AB^t$. Denote the first row of C by $(c_1, \dots, c_m)$ and observe that $c_1 = \sum_{i=1}^m a_ib_i$. Then

$$CC^t = AB^tBA^t = abI_m,$$

which immediately gives $ab = (a_1^2 + \cdots + a_m^2)(b_1^2 + \cdots + b_m^2) = c_1^2 + c_2^2 + \cdots + c_m^2$, where $c_1 = \sum_{i=1}^m a_ib_i$, as required. □

Note. This is another proof of the fact that a product of sums of 2^n squares is again a sum of 2^n squares, due to Knebusch and Scharlau.

We are now ready to prove the main theorem of the section. We shall require a lemma, proof of which will not be given here to avoid a lengthy interruption of the exposition.

Lemma 4.3. *Let F be a field such that $\underbrace{\langle 1, \dots, 1 \rangle}_{n}$ is anisotropic over F (that is, a nontrivial sum of squares is never zero in F). Then $t_1^2 + \cdots + t_n^2$ cannot be written as a sum of $n - 1$ squares in the rational function field $F(t_1, \dots, t_n)$.*

Proof. The Lemma follows from the Cassels-Pfister theorem and the Cassels representation theorem; we refer the reader to [7], Chapter 1 for a proof. □

Theorem 4.4 (Pfister)**.** *Let $m \geq 0$ and choose $n \in \mathbb{N}$ such that $2^m \leq n < 2^{m+1}$. Let $x_1, \dots, x_n$ be indeterminates over $\mathbb{R}$ and put $d = x_1^2 + \cdots + x_n^2$ in $k := \mathbb{R}(x_1, \dots, x_n)$. Then the field $F := k(\sqrt{-d})$ has level 2^m.*

Proof. Since

$$\left(\frac{x_1}{\sqrt{-d}}\right)^2 + \cdots + \left(\frac{x_{2^m}}{\sqrt{-d}}\right)^2 = -1$$

in F, we have $s(F) \leq n$. By Theorem 3.3 above, we must have $s(F) \leq 2^m$. Put $t = 2^m$ and suppose, if possible, that $s = s(F) < t$. Then we can find $\alpha_1, \ldots, \alpha_s \in F$ such that $\alpha_1^2 + \cdots + \alpha_s^2 = -1$. Put $\alpha_{s+1} = 1$ and $\alpha_{s+2} = \cdots = \alpha_t = 0$. Then

$$\alpha_1^2 + \cdots + \alpha_t^2 = 0.$$

Write $\alpha_i = a_i + b_i\sqrt{-d}$, where $a_i, b_i \in k$. Now, we must have $\sum_{i=1}^t b_i^2 \neq 0$. For, otherwise we would have $b_1 = \cdots = b_t = 0$, k being formally real; which in turn, would imply that $a_1 = \cdots = a_t = 0$, a contradiction. Since k is formally real and F is not, $\sqrt{-d} \notin k$. Thus,

$$\alpha_1^2 + \cdots + \alpha_t^2 = 0 \implies \sum_{i=1}^t a_i^2 - d\sum_{i=1}^t b_i^2 = 0 \text{ and } \sum_{i=1}^t a_i b_i = 0.$$

By Lemma 4.2, there exist $c_2, \ldots, c_t \in k$ such that

$$(a_1^2 + \cdots + a_t^2)(b_1^2 + \cdots + b_t^2) = \left(\sum_{i=1}^t a_i b_i\right)^2 + c_2^2 + \cdots + c_t^2$$

Therefore, putting $b = b_1^2 + \cdots + b_t^2$ and using the above two equations, we see that

$$\begin{aligned} d &= \frac{\sum_{i=1}^t a_i^2}{\sum_{i=1}^t b_i^2} \\ &= \frac{1}{b^2}\left(\sum_{i=1}^t a_i^2\right)\left(\sum_{i=1}^t b_i^2\right) \\ &= \frac{1}{b^2}\left[\left(\sum_{i=1}^t a_i b_i\right)^2 + c_2^2 + \cdots + c_t^2\right] \\ &= \left(\frac{c_2}{b}\right)^2 + \cdots + \left(\frac{c_t}{b}\right)^2, \end{aligned}$$

a sum of $t-1$ squares in k, contradicting Lemma 4.3. This proves the theorem.

□

5. T L C

The level of a commutative ring is defined in the same way as the level of a field.

Definition 5.1. *The level of a commutative ring A is defined to be the smallest positive integer n such that -1 is a sum of n squares of elements of A; we denote the level of A by $s(A)$.*

Theorem 5.2 (Dai, Lam, Peng)**.** *Given any positive integer n, there exists a commutative ring R_n such that $s(R_n) = n$.*

Proof. A natural chioce for R_n is $\mathbb{R}[x_1, \ldots, x_n]/(1 + x_1^2 + \cdots + x_n^2)$ [2]. Clearly, $s(R_n) \leq n$, since $-1 = x_1^2 + \cdots + x_n^2$ in R_n. Suppose, if possible, that $s(R_n) < n$. Then there exist polynomials $f_1(x), \ldots, f_{n-1}(x), p(x) \in \mathbb{R}[x]$, where x stands for $x_1, \ldots, x_n$ such that

[2]Note that R_n is, in fact, an integral domain, $1+x_1^2+\cdots+x_n^2$ being an irreducible polynomial in $\mathbb{R}[x_1, \ldots, x_n]$.

$$(*) \qquad -1 = f_1(x)^2 + \cdots + f_{n-1}(x)^2 + p(x)(1 + x_1^2 + \cdots + x_n^2)$$

in $\mathbb{R}[x_1, \ldots, x_n]$. We replace the variables x_k by ix_k ($i = \sqrt{-1}$) and write

$$f_k(ix) = g_k(x) + ih_k(x),$$

for all k, where $g_k(x), h_k(x) \in \mathbb{R}[x_1, \ldots, x_n]$ and

$$p(ix) = q(x) + ir(x),$$

where $q(x), r(x) \in \mathbb{R}[x_1, \ldots, x_n]$. Note that we have $h_k(-x) = -h_k(x)$ for all k. Putting ix_k for x_k in $(*)$ and comparing real parts of the resulting equation, we get

$$-1 = (g_1(x)^2 - h_1(x)^2) + \cdots + (g_{n-1}(x)^2 - h_{n-1}(x)^2) + q(x)(1 - x_1^2 - \cdots - x_n^2).$$

The polynomials $h_k(x)$ for $1 \leq k \leq n$ together define a function $\varphi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $\varphi(x) = (h_1(x), \ldots, h_{n-1}(x))$. Restrict φ to $S^{n-1} \hookrightarrow \mathbb{R}^n$. We now apply the Borsuk-Ulam theorem of topology to get a point $y = (y_1, \ldots, y_n) \in S^{n-1}$ such that $\varphi(y) = \varphi(-y)$. But we also have $\varphi(y) = -\varphi(-y)$, which implies $\varphi(y) = 0$. Therefore, $h_k(y) = 0$ for all k and we get an obvious contradiction

$$-1 = g_1(x)^2 + \cdots + g_{n-1}(x)^2,$$

since $y_1^2 + \cdots + y_n^2 = 1$ as y lies on the sphere S^{n-1}, proving the theorem.

$\square$

Note. Observe that the proof of Theorem 5.2 uses a topological fact, namely the Borsuk-Ulam theorem, which says that for any continuous map $\varphi : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $\varphi(x) = \varphi(-x)$. An equivalent formulation is the following: If $m < n$, then there exists no map $f : S^n \to S^m$ such that $f(-x) = -(f(x))$.

6. T L T S I

We shall consider pairs (X, i_X), where X is a topological space and i_X is an involution on X, that is, a continuous map $i_X : X \to X$ such that $i_X^2 = id_X$. By a morphism (or a *map*, for brevity) between pairs (X, i_X) and (Y, i_Y), we mean a continuous function $f : X \to Y$ which is *equivariant* with respect to involutions on X and Y : $f \circ i_X = i_Y \circ f$. Every topological space X can be considered as a pair (X, i_X), where $i_X = id_X$, and maps between such pairs are just the continuous functions between topological spaces. (In the language of category theory, the category of topological spaces can be considered as a full subcategory of the category of topological spaces with involution.) But we do not consider such pairs as we shall see in a moment that the only pairs (X, i_X) of interest are the ones where the involution is *fixed point-free*, that is, $i_X(x) \neq x$ for all $x \in X$.

Examples 6.1. (1) $X = \mathbb{R}$ or S^{n-1} and $i_X(x) = -x$.

(2) $X = \mathbb{C}^n$ and $i_X(z_1, \ldots, z_n) = (\overline{z_1}, \ldots, \overline{z_n})$, where $\overline{z_j}$ denotes the complex conjugate of z_j. Note that the set of fixed points of i_X is precisely $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$.

(3) Let $I \subseteq \mathbb{R}[x_1, \ldots, x_n]$ be an ideal. Let $X = V_{\mathbb{C}}(I) \subseteq \mathbb{C}^n$ be the affine algebraic set defined by I:

$$X = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid f(z_1, \ldots, z_n) = 0 \text{ for all } f \in I\},$$

with the *subspace topology induced from* $\mathbb{C}^n$. Note that the complex conjugation map $i_{\mathbb{C}^n}$ takes X to X; let i_X denote the restriction of $i_{\mathbb{C}^n}$ to X. Then (X, i_X) is topological space with involution. i_X is fixed point-free if and only if I has no real zeroes (for example, $I = (1 + x_1^2 + \cdots + x_n^2)$).

(4) Odd-dimensional real projective spaces $\mathbb{R}P^{2m-1}$ admit fixed point free involutions. Consider the map from $\mathbb{C}^m \setminus \{0\}$ to itself given by multiplication by the imaginary unit i: $(z_1, \ldots, z_m) \mapsto (iz_1, \ldots, iz_m)$. This induces a map from $\mathbb{R}^{2m} \setminus \{0\}$ to itself given by $(x_1, y_1, \ldots, x_m, y_m) \mapsto (-y_1, x_1, \ldots, -y_m, x_m)$. This, in turn, induces a map i_m from $\mathbb{R}P^{2m-1}$ to itself, which is an involution, since $[i^2(z_1, \ldots, z_n)] = [-z] = [z]$ in $\mathbb{R}P^{2m-1}$. On the other hand, it can be seen from algebraic topology (using Euler characteristic) that the even-dimensional projective spaces $\mathbb{R}P^{2m}$ do not admit fixed point-free involutions.

(5) The Stiefel manifold $V_{n,m}$ of orthogonal m-frames in $\mathbb{R}^n$, $m \leq n$, comes with a fixed point-free involution
$$(v_1, \ldots, v_r, v_{r+1}, \ldots, v_m) \mapsto (v_1, \ldots, v_r, -v_{r+1}, \ldots, -v_m),$$
where $r < m$ is fixed.

Notation. $(S^{n-1}, -)$ will denote the n-dimensional sphere with the antipodal involution $i_{S^{n-1}}(x) = -x$.

Definition 6.2. *The level of a topological space with involution* (X, i_X) *is defined to be the smallest positive integer n such that there exists a map* $(X, i_X) \to (S^{n-1}, -)$ *of pairs. If no such n exists, then we define the level to be* ∞. *We denote the level of* (X, i_X) *by* $s(X, i_X)$.
$$s(X, i_X) = \inf \{n \in \mathbb{N} \mid \text{ there exists a map of pairs } f : (X, i_X) \to (S^{n-1}, -)\}.$$

Remark. The level $s(X, i_X)$ depends also on the involution i_X. For example, the Stiefel manifold $V_{n,m}$ has two basic fixed-point free involutions:
$$\delta : (v_1, \ldots, v_m) \mapsto (-v_1, \ldots, -v_m),$$
$$\epsilon : (v_1, \ldots, v_m) \mapsto (v_1, \ldots, v_{m-1}, -v_m).$$
Dai and Lam in [3] have proved that
$$s(V_{n,2}, \delta) = n, \text{ for all } n \geq 2,$$
whereas
$$s(V_{n,2}, \epsilon) = \begin{cases} n-1 & \text{if } n = 2, 4, 8\ , \\ n & \text{otherwise.} \end{cases}$$

Examples 6.3. (1) If i_X has a fixed point $x \in X$, then $s(X, i_X) = \infty$. For, otherwise there would exist $n \in \mathbb{N}$ and a map $f : (X, i_X) \to (S^{n-1}, -)$ of pairs. But then $f(x) \in S^{n-1}$ would be fixed by the antipodal map, a contradiction.

(2) If there exists a map of pairs $g : (X, i_X) \to (Y, i_Y)$, then $s(X, i_X) \leq s(Y, i_Y)$; for, any map of pairs $f : (Y, i_Y) \to (S^{n-1}, -)$ gives a map of pairs $f \circ g : (X, i_X) \to (S^{n-1}, -)$. In particular, if there exist maps of pairs $(X, i_X) \to (Y, i_Y)$ and $(Y, i_Y) \to (X, i_X)$, then $s(X, i_X) = s(Y, i_Y)$.

Proposition 6.4. *Let* (X, i_X) *be a topological space with involution, where the involution* i_X *is fixed point-free.*

(1) *If* X *is a topological subspace of* $\mathbb{R}^n$, *then* $s(X) \leq n$.

(2) *If X is a topological subspace of $\mathbb{C}^n$ and i_X is the restriction of the complex conjugation on $\mathbb{C}^n$ to X, then $s(X) \leq n$.*

Proof. (1) This follows by defining $f : (X, i_X) \to (S^{n-1}, -)$ by

$$f(x) := \frac{x - i_X(x)}{\|x - i_X(x)\|}, \text{ for all } x \in X.$$

(2) This follows by defining $f : (X, i_X) \to (S^{n-1}, -)$ by

$$f(z_1, \dots, z_n) := \left(\frac{y_1}{d}, \cdots, \frac{y_n}{d}\right),$$

where $z_j = x_j + iy_j$ and $d = \sqrt{y_1^2 + \cdots + y_n^2}$.

□

Do there exist topological spaces with any prescribed level? The answer is affirmative and examples are provided by spheres with antipodal involution. This is the most vital result in the theory of topological level.

Theorem 6.5. *$s(S^{n-1}, -) = n$ for all $n \in \mathbb{N}$.*

Proof. This is just a disguised version of the Borsuk-Ulam theorem in topology: there is no map $f : S^{n-1} \to S^{m-1}$ with $f(-x) = -(f(x))$ for $m < n$. Clearly, $id_{S^{n-1}}$ is a map of pairs $(S^{n-1}, -) \to (S^{n-1}, -)$. Hence, $s(S^{n-1}, -) = n$ for all $n \in \mathbb{N}$. □

Relation between Level of Topological Spaces with Involution and Level of Rings. The topological level of pairs (X, i_X) and the algebraic level of rings are intimately related. We can associate an $\mathbb{R}$-algebra of complex-valued functions to any topological space with involution (X, i_X) as follows:. We define

$$A_{(X,i_X)} := \{f \mid f : (X, i_X) \to (\mathbb{C}, \bar{.}) \text{ is a map of pairs }\},$$

where $\bar{.}$ denotes the involution $z \mapsto \bar{z}$ on $\mathbb{C}$. $A_{(X,i_X)}$ clearly contains the constant real-valued functions on X. Further, for any $f, g \in A_{(X,i_X)}$, their pointwise addition $f + g$ and pointwise multiplication $f \cdot g$ are in $A_{(X,i_X)}$. It is easy to check that $A_{(X,i_X)}$ has an $\mathbb{R}$-algebra structure with the homomorphism $\mathbb{R} \to A_{(X,i_X)}$ given by

$$a \longmapsto \text{ the constant function } a \text{ on } X.$$

Let $g : (X, i_X) \to (Y, i_Y)$ be a map between two topological spaces with involution. Then g induces an $\mathbb{R}$-algebra homomorphism $g^* : A_{(Y,i_Y)} \to A_{(X,i_X)}$ given by

$$g^*(f) = f \circ g,$$

for all $f \in A_{(Y,i_Y)}$ (note that $g^*(1_Y) = 1_Y \circ g = 1_X$). in the language of category theory, we can express this fact by saying that

$$(X, i_X) \rightsquigarrow A_{(X,i_X)}$$
$$g \rightsquigarrow g^*$$

gives a contravariant functor from the category of topological spaces with involution to the category of commutative $\mathbb{R}$-algebras.

We are now ready to prove the main theorem of the section.

Theorem 6.6 (Dai, Lam)**.** *The level of any topological space with involution* (X, i_X) *equals the level of* $A_{(X,i_X)}$ *as a commutative ring.*

$$s(X, i_X) = s(A_{(X,i_X)}).$$

Proof. We break down the proof in three steps

Step 1. Let $A_n := A_{(S^{n-1},-)}$*; then* $s(A_n) \le n$*.*

Let $x = (x_1, \ldots, x_n) \in S^{n-1}$; then $x_1^2 + \cdots + x_n^2 = 1$. Consider the functions $f_j(x) := ix_j$, for $j = 1, \ldots, n$ on S^{n-1}, where $i = \sqrt{-1}$. Clearly, $f_j \in A_n$ and $(f_1^2 + \cdots + f_n^2)(x) = -1(x_1^2 + \cdots + x_n^2) = -1$, so $s(A_n) \le n$.

Step 2. For any pair (X, i_X)*,* $s(A_{(X,i_X)}) \le s(X, i_X)$*.*

We may assume that $s(X, i_X) = n$ is finite; for, thee is nothing to prove when $s(X, i_x) = \infty$. Then there exists a map of pairs $g : (X, i_X) \to (S^{n-1}, -)$. Let $g^* : A_n \to A_{(X,i_X)}$ be the homomorphism induced by g. From Step 1 we have $f_1^2 + \cdots + f_n^2 = -1$ in A_n. Applying g^* we see that

$$g^*(f_1)^2 + \cdots + g^*(f_n)^2 = -1$$

in $A_{(X,i_X)}$, which implies that $s(A_{(X,i_X)}) \le n = s(X, i_X)$.

Step 3. $s(X, i_X) \le s(A_{(X,i_X)})$.

Again, we may assume that $s(A_{(X,i_X)}) = n$ is finite. Then there exist $f_1, \ldots, f_n \in A_{(X,i_X)}$ with

$$f_1^2 + \cdots + f_n^2 = -1$$

Put $f_j = p_j + iq_j$ for $j = 1, \ldots, n$, where p_j, q_j are real-valued continuous functions on X. If q_j's have a common zero $x \in X$, then $p_1(x)^2 + \cdots + p_n(x)^2 = -1$ in $\mathbb{R}$, a contradiction. Hence, q_j's do not vanish simultaneously on X. We can thus define a continuous map $h : X \to S^{n-1}$ by

$$h(x) = \frac{1}{\sqrt{q_1(x)^2 + \cdots + q_n(x)^2}} \left(q_1(x), \ldots, q_n(x)\right).$$

In $A_{(X,i_X)}$, we have

$$f_j(i_X(x)) = \overline{f_j(x)} = p_j(x) - iq_j(x),$$

whence $q_j(i_X(x)) = -q_j(x)$ for all j. This shows that h is a map of pairs $(X, i_X) \to (S^{n-1}, -)$. Therefore, $s(X, i_X) \le n = s(A_{(X,i_X)})$, completing the proof.

□

Remark. In particular, we have $s(A_n) = n$.

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