Synopsis of the Ph.D. thesis titled
“Some problems in combinatorics”

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In this thesis we study three combinatorial problems motivated by questions in design theory and graph theory. We now give the motivation for each of these problems and a brief description of the results.

A \textit{k-class association scheme} consists of a set \( X \) and \( k + 1 \) nonempty symmetric binary relations \( R_0, R_1, \ldots, R_k \) satisfying the following conditions.

(a) \( R_0 = \{(x, x) \mid x \in X\} \);

(b) Given any \( (x, y) \in R_\ell \), there are exactly \( P_{ij}^\ell \) elements \( z \in X \) such that \( (x, z) \in R_i \) and \( (z, y) \in R_j \).

One of the main examples of the above notion is the \textit{Johnson scheme}: The point set \( X \) is the set of all \( k \)-element subsets of a set \( S \). Two \( k \)-sets \( A \) and \( B \) are declared to be \( i \)-th associates, i.e., \( (A, B) \in R_i \), when \( |A \cap B| = k - i \). It is easy to see that the \( 0 \)-th associates are

\[ \{(A, A) : |A| = k \text{ and } A \subseteq S\} \]

and the numbers \( P_{ij}^\ell \) exist by symmetry.

For each \( R_i \) of an association scheme the corresponding \textit{association matrix} (adjacency matrix) is defined by

\[ A_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_i; \\ 0 & \text{otherwise.} \end{cases} \]

Let \( \mathcal{A} \) denote the linear span of \( A_0, A_1, \ldots, A_k \) over \( \mathbb{R} \). Note that the matrices \( A_i \) are independent and closed under matrix multiplication. The resulting algebra \( \mathcal{A} \) is called the \textit{Bose-Mesner Algebra} of the scheme.

\textbf{Theorem 0.1} There exists an orthogonal decomposition \( \mathbb{R}^X = V_0 \oplus V_1 \oplus \cdots \oplus V_k \) such that the orthogonal projections \( E_0, E_1, \ldots, E_k \) is a basis for \( \mathcal{A} \).

For the Johnson scheme the orthogonal decomposition can be constructed in a canonical way. Let \( M_{ik} \) denote the \( \binom{n}{i} \times \binom{n}{k} \) matrix whose rows and columns are indexed by \( i \)-subsets and \( k \)-subsets of \( S \) and

\[ M_{ik}(A, B) = \begin{cases} 1 & \text{if } A \subseteq B; \\ 0 & \text{otherwise.} \end{cases} \]

Let \( U_i \) denote the row space of \( M_{ik} \). The \( V_i \)'s are canonically described by the following lemmas.

\textbf{Lemma 0.2} \( U_{i-1} \subseteq U_i \) for \( 1 \leq i \leq k \) and \( \dim U_i = \binom{n}{i} \).

\textbf{Lemma 0.3} \( U_i = V_i \perp U_{i-1} \) for \( 1 \leq i \leq k \).
A vector \( \omega \in \mathbb{R}^X \) is said to be a general vector iff \( (\omega, \chi_x) \neq 0 \), for all \( x \in X \) (here, \( (\ , \ ) \) denotes innerproduct). The \( i^{th} \) distribution invariant is defined as

\[
vt_i(X) = \min_{\omega} |\{ x \mid x \in X, \omega \in V_i, (\omega, \chi_x) > 0 \text{ and } \omega \text{ general}\}|.
\]

The distribution invariants were first introduced by Thomas Bier [5] while attempting to answer certain problems in topology. He also proposed the following problem.

**Problem 0.4** Find all the distribution invariants for the known classical association schemes.

In [11] the following conjecture was made.

**Conjecture 0.5** (Manickam-Singhi [11]) For the Johnson scheme \( J(n, k) \),

\[
vt_1(J(n, k)) = \binom{n-1}{k-1} \text{ if } n \geq 4k.
\]

In [11] it was observed that Conjecture 0.5 reduces to the following one. For a nonnegative integer \( k \), we denote the set \( \{1, 2, \ldots, k\} \) by \([k]\).

**Conjecture 0.6** (Manickam-Singhi [11]) Let \( x : [n] \rightarrow \mathbb{R} \) be a map such that \( \sum_{i \in [n]} x_i \geq 0 \). (For any \( i \), its image is denoted by \( x(i) \).) Let \( \mathcal{F} = \{ J \subset [n] : |J| = k, \text{ and } \sum_{j \in J} x(j) \geq 0 \} \). Then \( |\mathcal{F}| \geq \binom{n-1}{k-1} \) whenever \( n \geq 4k \).

For any two integers \( r \) and \( \ell \) let \( [r]_\ell \) denote the smallest positive integer congruent to \( r \) (mod \( \ell \)). In [11] Manickam and Singhi showed that the conclusion of the conjecture holds when \( k \) divides \( n \).

In [6] Bier and Manickam have shown that if \( k \geq 3 \) and \( n = 3k + 1 \), then the conjecture does not hold. The main result of that paper is the following: If \( k > 3 \) and \( n \geq k(k-1)^2(k-2)^k + k(k-1)^2(k-2) + k[k]_k \), then the conjecture holds.

Srinivasan in [14] has pointed out an important application of this conjecture; he has shown that the validity of this conjecture settles some special cases of a number-theoretic conjecture by Alladi, Erdös and Vaaler on multiplicative functions [1].

In [12] this conjecture has been settled for \( k = 3 \). If \( |\{ i \in [n] : x(i) \geq 0 \}| \leq \frac{n}{2} \), then also the conjecture holds [7].

In Chapter 1 we give a short proof to show that the conjecture holds for every fixed \( k \) and \( n \) sufficiently large; the bound given in Chapter 1 is smaller than that found in [6]. The concerned result is the following.
Theorem 0.7 ([3]) Let \( x \) and \( F \) be defined as above. Then \(|F| \geq \binom{n-1}{k-1}\), if \( n \geq 2^{k+1}e^k k^{k+1} \).

In Chapter 2 we consider a problem in root systems. Let \( \mathbb{R} \) and \( \mathbb{Z} \) denote the set of reals and the set of integers respectively and \( \mathbb{E} \) be a finite dimensional vector space over \( \mathbb{R} \) with innerproduct \((\ ,\ )\). If \( v_0, v_1, v_2, \ldots, v_n \) are vectors in \( \mathbb{E} \) such that \( v_0 = \sum_{i=1}^{n} t_i v_i \) where \( t_i \in \mathbb{Z} \) for \( i = 1, \ldots, n \), then \( v_0 \) is called an integral combination of \( v_1, v_2, \ldots, v_n \). Let \( S \) and \( T \) be two subsets of \( \mathbb{E} \). If every vector of \( S \) is an integral combination of vectors in \( T \), then we say that \( S \) is generated by \( T \).

Let \( S \) be any subset of \( \mathbb{E} \). We can associate a simple graph with \( S \) as follows: Its vertex set is \( S \) and two vertices are joined if their innerproduct is nonzero. If this graph is connected, then \( S \) is called indecomposable; otherwise it is decomposable. Note that when \( S \) is decomposable, it has a proper nonempty subset \( T \) such that for all \( x \in T \) and for all \( y \in S \setminus T \), \((x, y) = 0\).

**Definition 0.8** A subset \( \Phi \) of \( \mathbb{E} \) is called a root system in \( \mathbb{E} \) if the following axioms are satisfied:

\begin{enumerate}
\item[(R1)] \( \Phi \) is finite and spans \( \mathbb{E} \); it does not contain 0.
\item[(R2)] If \( \alpha \in \Phi \) and \( t \in \mathbb{R} \), then \( t\alpha \in \Phi \) only when \( t = \pm 1 \).
\item[(R3)] If \( \alpha, \beta \in \Phi \), then \( \frac{2(\alpha, \beta)}{(\beta, \beta)} \) is an integer.
\item[(R4)] For any \( \alpha, \beta \in \Phi \), \( \alpha - \frac{2(\alpha, \beta)}{(\beta, \beta)} \beta \in \Phi \).
\end{enumerate}

The elements of \( \Phi \) are called roots. For any \( \alpha \in \Phi \), \( \|\alpha\| \) is called the length of \( \alpha \). When \( \Phi \) is indecomposable (decomposable) it is also called irreducible (reducible).

Henceforth, \( \Phi \) stands for a root system. For any \( \alpha, \beta \in \Phi \), we denote the integer \( \frac{2(\alpha, \beta)}{(\beta, \beta)} \) by \( \langle \alpha, \beta \rangle \).

**Definition 0.9** A subset \( \Delta \) of \( \Phi \) is called a base if the following conditions are satisfied.

\begin{enumerate}
\item[(B1)] \( \Delta \) is a basis of \( \mathbb{E} \).
\item[(B2)] Each root \( \beta \) can be written as \( \sum_{\alpha \in \Delta} k_\alpha \alpha \) where either all the \( k_\alpha \)'s are nonnegative integers or all of them are nonpositive integers.
\end{enumerate}

Bases play a vital role in classifying root systems. It has been shown that a root system is completely determined by a base. If a root system \( \Phi \) has two bases, then there exists an automorphism of \( \mathbb{E} \) fixing \( \Phi \) and taking one base into another.
A natural question to ask in conjunction with (B2) is the following: If $S$ is a linearly dependent subset of $\Phi$, can there be a linearly independent subset of $S$ which generates $S$? We answer this question affirmatively in Chapter 2:

**Theorem 0.10 ([4])** Any indecomposable subset $S$ of $\Phi$ is generated by an indecomposable linearly independent subset of $S$.

The above theorem for any irreducible root system of unique root length has been proved in [15].

A subset $S$ of $\Phi$ is called *obtuse* if for all distinct $\alpha, \beta \in S$, $(\alpha, \beta) \leq 0$; in other words, $S$ is obtuse if the angle between any two different roots of $S$ is not acute. It can be shown that any base of a root system is obtuse. (See [8, p. 47].)

**Definition 0.11** A nonempty subset $C$ of $\Phi$ is called a *circuit* if it satisfies the following:

1. $C$ is obtuse.

2. $C$ is a minimal linearly dependent set; i.e., $C$ is linearly dependent but every proper subset of $C$ is linearly independent.

3. If a root $\alpha \in C$, then $-\alpha \notin C$.

Note that by minimality, any circuit is indecomposable. By (3) and linear dependence, any circuit has at least three roots.

Circuits have been classified in [9]. In the process of proving the main result of this chapter, we also give, for the sake of completeness, a method of classification which is different from that in [9].

We now discuss the problem studied in Chapter 3. Consider the following basic question. Let $n$, $t$, $k$ be three positive integers such that $t < k < \frac{n}{2}$. Let $f : \binom{[n]}{k} \to \mathbb{Z}_{\geq 0}$. Let $\partial_t f : \binom{[n]}{t} \to \mathbb{Z}_{\geq 0}$ be defined by

$$
\partial_t f(T) = \sum_{T \subset B, B \in \binom{[n]}{k}} f(B)
$$

for all sets $T$ in $\binom{[n]}{t}$.

**Problem 0.12** (General $(t,k)$–existence problem)

a) Characterize all nonnegative functions $g : \binom{[n]}{t} \to \mathbb{Z}_{\geq 0}$ such that there exists a function $f : \binom{[n]}{k} \to \mathbb{Z}_{\geq 0}$ satisfying $\partial_t f = g$.

b) Characterize all nonnegative functions $g : \binom{[n]}{t} \to \mathbb{Z}_{\geq 0}$ such that there exists a function $f : \binom{[n]}{k} \to \{0,1\}$ satisfying $\partial_t f = g$. 
Some special cases like $t$-designs, degree sequences, partial Steiner systems and some related problems like $f$-vectors of simplicial complexes have received a lot of attention during the last three decades. Yet we know very little about the problem.

This work is motivated by the long standing open problem of characterizing the degree sequences of simple $r$-uniform hypergraphs, which is a special case of Problem 0.12 b). For $r = 2$ there are well known characterizations by “Havel and Hakimi” and “Erdős and Gallai”.

If $r > 2$ we do not even know whether the problem is NP-complete or not.

In [2] the convex hull of all degree sequences of simple $r$-uniform hypergraphs was studied. The extreme points of this polytope were shown to be the degree sequences of threshold hypergraphs. The starting point of this chapter is an alternative characterization of threshold hypergraphs. We now give precise definitions.

An $r$-uniform hypergraph is a triple $H = (V, E, \phi)$, where $V$ is a finite set of vertices, $E$ is a finite set of edges, and

\[ \phi : E \to \binom{V}{r} \]  

(the set of all $r$-element subsets of $V$).

Note that this definition allows parallel edges. The degree $d(v)$ of a vertex $v$ is $\# \{ e \in E : v \in \phi(e) \}$ and $d_H = d = (d(v) : v \in V)$ is the degree sequence of $H$.

The hypergraph $H$ is simple if there are no parallel edges, i.e., $\phi$ is injective. In this case we identify $E$ with the image of $\phi$ and take $E$ to be a subset of $\binom{V}{r}$. An $r$-uniform hypergraph will be called a $r$-graph and a 2-graph will be called a graph. An $r$-graphical sequence is the degree sequence of a simple $r$-graph and a 2-graphical sequence will be called a graphical sequence.

A simple $r$-graph $(V, E)$ is said to be threshold if there are real weights $c(v) \in \mathbb{R}$, $v \in V$ such that, for all $X \in \binom{V}{r}$, we have (using the notation $c(X) = \sum_{v \in X} c(v)$)

\[
X \in E \implies c(X) > 0, \\
X \not\in E \implies c(X) < 0.
\]

The degree sequence of a threshold $r$-graph will be called a $r$-threshold sequence. A 2-threshold sequence will be called a threshold sequence.

Let $H = (V, E, \phi)$ be a $r$-graph. Assume that the edges of $H$ are colored red or blue, the coloring being given by $C : E \to \{ R, B \}$. Consider the real vector space $\mathbb{R}^E$, with coordinates indexed by the set of edges of $H$. We write an element $x \in \mathbb{R}^E$ as $x = (x(e) : e \in E)$. For a subset $F \subseteq E$ and $v \in V$, by $F(v)$ we mean the set of all edges in $F$ containing the vertex $v$, i.e., $F(v) = \{ e \in F : v \in \phi(e) \}$. For a subset $F \subseteq E$, $F_R$ denotes the set of red edges in $F$. Similarly we define $F_B$. For an edge $e \in E$, the characteristic vector $\chi(e) \in \mathbb{R}^E$ is defined by

\[
\chi(e)(f) = \begin{cases} 
1, & \text{if } f = e; \\
0, & \text{if } f \neq e.
\end{cases}
\]
The red degree \( r(v) \) of a vertex \( v \in V \) is the number of red edges containing the vertex \( v \), i.e., \( r(v) = \# \{ e \in E : v \in \phi(e) \text{ and } C(e) = R \} \). Similarly, we define the blue degree \( b(v) \) of a vertex \( v \in V \).

The alternating cone of a 2-colored \( r \)-graph \( H \), denoted \( \mathcal{A}(H, C) \) (when the coloring \( C \) is understood, we suppress it from the notation and write \( \mathcal{A}_2(H) \)), is defined to be the set of all elements \( x = (x(e) : e \in E) \) in \( \mathbb{R}^E \) satisfying the following system of homogeneous linear inequalities:

\[
\begin{align*}
\sum_{e \in E_R(v)} x(e) - \sum_{e \in E_B(v)} x(e) &= 0, \quad v \in V, \\
x(e) &\geq 0, \quad e \in E.
\end{align*}
\]

We refer to (3) as the balance condition at vertex \( v \).

Let \( H = (V, E) \) be a simple \( r \)-graph. The 2-colored simple \( r \)-graphs associated to \( H \) is defined by \( \bar{H} = (V, \binom{V}{2}) \), where \( e \in \binom{V}{2} \) is colored red if \( e \in E \) and colored blue if \( e \notin E \). Our main motivation for defining the alternating cone is the following observation.

**Theorem 0.13** Let \( H = (V, E) \) be a simple \( r \)-graph. Then \( H \) is threshold if and only if \( \dim \mathcal{A}(\bar{H}) = 0 \).

In [2] it was shown that \( r \)-threshold sequences are uniquely realizable, i.e., if \( d = (d(v) : v \in V) \) is a \( r \)-threshold sequence and \( H = (V, E) \) and \( H' = (V, E') \) are two simple \( r \)-graphs with degree sequence \( d \), then \( E = E' \). It follows that the property of a simple \( r \)-graph \( H \) being threshold (i.e., \( \dim \mathcal{A}(\bar{H}) = 0 \)) depends only on the degree sequence of \( H \). We show that this result continues to hold for higher dimensions of the alternating cone. In fact we prove the following theorem.

**Theorem 0.14** Let \( H = (V, E, \phi) \) be a \( r \)-graph and let \( C_1 : E \to \{ R, B \} \) and \( C_2 : E \to \{ R, B \} \) be 2-colorings. Denote the red and blue degrees of \( v \in V \) under \( C_i \) by \( r_i(v) \) and \( b_i(v) \) respectively, \( i = 1, 2 \). Assume \( r_1(v) = r_2(v) \), for all \( v \in V \). Then \( \dim \mathcal{A}(H, C_1) = \dim \mathcal{A}(H, C_2) \).

Next we relate the dimension of the alternating cone to the concept of majorization. We begin with a few definitions.

We say a simple \( r \)-graph \( (V, E) \) is an ideal (called a 1-ideal in [2]) if there is a linear order \( < \) on \( V \) such that, for all \( X \subseteq E, i \in X, \) and \( j < i \) we have \( j \in X \) or \( (X - \{ i \}) \cup \{ j \} \subseteq E \). The degree sequence of an ideal \( r \)-graph will be called an \( r \)-ideal sequence. It is well known that \( 2 \)-threshold and \( 2 \)-ideal sequences coincide (see [10]) but for \( r \geq 3 \) the set of \( r \)-threshold sequences is a proper subset of the set of \( r \)-ideal sequences. An example with \( r = 3 \) is given in [2].

Let \( a = (a(1), \ldots, a(n)) \) and \( b = (b(1), \ldots, b(n)) \) be real sequences of length \( n \). Denote the \( i \)-th largest component of \( a \) (respectively, \( b \)) by \( a[i] \)
(respectively, \(b[i]\)). We say that \(a\) majorizes \(b\), denoted \(a \succeq b\), if
\[
\sum_{i=1}^{k} a[i] \geq \sum_{i=1}^{k} b[i], \quad k = 1, \ldots, n,
\]
with equality for \(k = n\). The majorization is strict, denoted \(a \succ b\), if at least one of the inequalities is strict, namely if \(a\) is not a permutation of \(b\).

In Chapter 3 of [10], eight different characterizations of graphical sequences are given. The last of these characterizations states that a nonnegative integral sequence is a graphical sequence if and only if it is majorized by a threshold sequence. We now give a hypergraph generalization of this result.

**Theorem 0.15** A nonnegative integral sequence is a \(r\)-graphical sequence if and only if it is majorized by an \(r\)-ideal sequence.

**Theorem 0.16** Let \(H_1 = ([n], E_1)\) and \(H_2 = ([n], E_2)\) be two simple \(r\)-graphs with degree sequences \(d_1 = (d_1(v) : v \in [n])\) and \(d_2 = (d_2(v) : v \in [n])\) respectively. Suppose \(d_1 \succeq d_2\). Then \(\dim A(H_1) \leq \dim A(H_2)\).

The last result has the following interpretation in the case of simple graphs: Let \(d\) be a graphical sequence. Then there is a threshold sequence \(e\) that majorizes \(d\) and there are simple graphs \(H_0, H_1, H_2, \ldots, H_l\) such that \(d_{H_i}\) majorizes \(d_{H_{i-1}}\), for \(i = 1, \ldots, l\) and \(d_{H_0} = d, \ d_{H_l} = e\). We then have \(\dim A(H_0) \geq \dim A(H_1) \geq \cdots \geq \dim A(H_l) = 0\). Thus \(\dim A(H_0)\) is a kind of measure of how nonthreshold the sequence \(d\) is.

In order to obtain further results on the alternating cone, we restrict attention to the case of graphs. We begin with a few definitions.

A walk in \(G\) is a sequence
\[
W = (v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_m), \quad m \geq 0,
\]
where \(v_i \in V\) for all \(i\); \(e_j \in E\) for all \(j\); and \(e_j\) has endpoints \(v_{j-1}\) and \(v_j\), for all \(j\) (i.e., \(\phi(e_j) = \{v_{j-1}, v_j\}\)). We say that \(W\) is a \(v_0 - v_m\) walk of length \(m\). We call \(e_1\) the first edge of \(W\) and \(e_m\) the last edge of \(W\). A walk in which all the edges are distinct is said to be a trail. A path is a walk in which all the vertices are distinct. If \(m \geq 1\) and all \(v_i\)'s are distinct except \(v_0 = v_m\), then the walk is called a cycle.

The walk \(W\) is said to be closed if \(v_0 = v_m\). The walk \(W\) is said to be internaltially alternating if, for all \(1 \leq j \leq m - 1\), the edges \(e_j\) and \(e_{j+1}\) have different colors. The walk \(W\) is said to be alternating if \(W\) is internally alternating and \(W\) closed implies that \(e_1\) and \(e_m\) also have different colors. Note that a walk can be closed and internally alternating without being alternating. However, if it is known that the starting and final vertices (i.e., \(v_0\) and \(v_m\)) are distinct then there is no difference between internally alternating
and alternating walks and we shall use the word alternating in this case. A closed alternating walk (respectively, closed alternating trail) is denoted a CAW (respectively, a CAT). The characteristic vector of a walk $W$ as in (5) above is defined to be $\chi(W) = \sum_{i=1}^{m} \chi(e_i)$.

An even alternating cycle in a 2-colored graph is a subgraph $C$ which is a cycle such that the two edges incident with any vertex of $C$ have different colors. See Figure 1. Walking once around a even alternating cycle $C$ gives a CAW whose characteristic vector is 1 on the edges of $C$ and zero outside.

An even alternating cycle will also be called simply an alternating cycle.

An odd alternating cycle with base $v$ in a 2-colored graph is a subgraph $C$ which is a cycle containing $v$, has an odd number of edges, and whose edges alternate in color except at the vertex $v$, where the two incident edges have the same color.

Now we define an alternating bicycle. We start with an alternating path $P$ (i.e., the two edges incident at every internal vertex of $P$ have different colors). Let the end vertices of $P$ be $u$ and $v$. We consider two cases:

(i) $u \neq v$: Take two vertex disjoint odd alternating cycles $C_1$ and $C_2$ with bases $u$ and $v$ respectively, such that no internal vertex of $P$ is on either $C_1$ or $C_2$ and such that the (unique) edge of $P$ incident at $u$ (respectively, $v$) has color different from the common color of the two edges of $C_1$ (respectively, $C_2$) incident at $u$ (respectively, $v$).

(ii) $u = v$: Take two odd alternating cycles $C_1$ and $C_2$ with base $u$ such that $C_1$ and $C_2$ have no other vertices (apart from $u$) in common and such that the common color of the two edges of $C_1$ incident at $u$ is different from the common color of the two edges of $C_2$ incident at $u$.

The subgraph $C_1 \cup C_2 \cup P$ (in case (ii) $P$ is empty) is called an alternating bicycle. See Figure 2.

This alternating bicycle defines the following CAW $W$: start at $u$, go around $C_1$, then traverse $P$ from $u$ to $v$, go around $C_2$, then traverse $P$ from $v$ to $u$. Clearly, $\chi(W)$ is a $\{0, 1, 2\}$ valued vector defined by $\chi(W) = \chi(C_1) + 2\chi(P) + \chi(C_2)$.

A CAW $W$ is said to be irreducible if $\chi(W)$ cannot be written as $\chi(W_1) + \chi(W_2)$, for CAW’s $W_1$ and $W_2$. For instance, the CAW’s coming from alter-
nating cycles and bicycles are easily seen to be irreducible. Similarly A CAT $T$ is said to be irreducible if $\chi(T)$ cannot be written as $\chi(T_1) + \chi(T_2)$, for CAT’s $T_1$ and $T_2$.

**Theorem 0.17** (i) The extreme rays of the cone $\mathcal{A}(G, C)$ are the characteristic vectors of the CAW’s determined by the alternating cycles and bicycles in $G$.

(ii) Every integral vector in the alternating cone is a nonnegative integral combination of the characteristic vectors of irreducible CAW’s.

(iii) Every integral $\{0,1\}$ vector in the alternating cone is a nonnegative integral combination of the characteristic vectors of irreducible CAT’s.

Let $G(V, E, \phi)$ be 2-colored graph (with colors red and blue) with red degree sequence $d_R$ and blue degree sequence $d_B$. Let $E' = \{ e \in E \mid \text{a closed alternating walk exists through } e \}$. We have the following observation.

**Lemma 0.18** The set $E'$ depends only on the degree sequence $d_R$ (or, equivalently, $d_B$) and not on the particular realization.

Consider the sub-graph $G'(V, E')$ of $G(V, E)$. Let the number of bipartite components in $G'(V, E')$ be $b$. We give the following formula for the dimension of the alternating cone, which is similar to the formula for the dimension of the cycle space of an oriented graph.

**Theorem 0.19** The dimension of the alternating cone $\mathcal{A}(\hat{G}, C)$ is given by $\#E' - \#V + b$.

Note that the theorem gives a polynomial time algorithm to determine the dimension of the alternating cone. Namely we can find $E'$ by using matrix multiplication techniques. Then by graph search algorithms we can compute $b$. This raises the following question.

**Question 0.20** Given a simple graph $G$ with degree sequence $d_G$, is there an algorithm which works only with $d_G$ and computes the dimension of $\mathcal{A}(G)$?

The special case of asking when dimension of $\mathcal{A}(\hat{G}) = 0$ has a well known combinatorial answer (see [10]).

Motivated by Hoffman’s circulation theorem for directed graphs we consider the following question.

**Question 0.21** Let $G = (V, E, \phi)$ be a 2-colored graph. Let the colors be red and blue. Let $u, \ell : E \to \mathbb{Z}^+$ satisfy $u \geq \ell \geq 0$. Does there exist a $p : E \to \mathbb{Z}^+$ such that $u \geq p \geq \ell$ and

$$\sum_{e \in E_R(v)} p(e) = \sum_{e \in E_B(v)} p(e), \text{ for all } v \in V?$$
In the remaining part of this section we state the results necessary to
answer the above question. We begin with a few definitions.

We say a function $f$ (or flow $f$) is balanced if it satisfies the balance
condition. If a flow $f$ satisfies $\ell \leq f \leq u$ we say it is feasible. If we can find a
flow which is balanced and feasible then we have solved the problem. Let us
assume that we have an $f$ which is balanced but not feasible. Given a graph
$G(V, E, \phi)$ with coloring $C$ and an integral flow $f$ we construct a residual
graph $G_r(V, E', \phi')$ with coloring $C'$ as follows.

For all $e \in E$ we do the following.
Case 1) $f(e) < u(e) - 1$: Put edges $e^1, e^2 \in E'$ with $\phi'(e^1) = \phi'(e^2) = \phi(e)$
and $C'(e^1) = C'(e^2) = C(e)$.
Case 2) $f(e) = u(e) - 1$: Put edge $e^1 \in E'$ with $\phi'(e^1) = \phi(e)$ and $C'(e^1) = C(e)$.
Case 3) $f(e) > \ell(e) + 1$: Put edges $e^1, e^2 \in E'$ with $\phi'(e^1) = \phi'(e^2) = \phi(e)$
and $C'(e^1) = C'(e^2) = \text{the unique element of } \{\text{Red, Blue}\} - C(e)$.
Case 4) $f(e) = \ell(e) + 1$: Put edge $e^1 \in E'$ with $\phi'(e^1) = \phi(e)$ and $C'(e^1) = \text{the unique element of } \{\text{Red, Blue}\} - C(e)$.

We have the following observation.

**Lemma 0.22** Suppose there exists a balanced and feasible flow and let $f$ be
any balanced flow which is not feasible and $f \leq u$. Then the following hold.
1) There exists an edge $e \in E$ with $f(e) < \ell(e)$ and an edge $e'$ in $G_r(V, E', \phi')$
with $\phi'(e') = \phi(e)$ and $C'(e') = C(e)$.
2) There exists an closed alternating trail in the residual graph containing
the edge $e'$.

This suggests an augmenting path type algorithm to construct a feasible
and balanced flow if it exists.

This leads us to the following problem, which we call the alternating
reachability problem: Given a set $S \subseteq V$, either find an alternating $s - t$
trail for some $s, t \in S$, $s \neq t$ or show that no such trail exists.

Suppose we have a simple graph and a matching in it. Take $S$ to be the
set of exposed vertices. Color the edges in the matching red and the edges
not in the matching blue. In this situation, the problem above is the same
as that of finding an augmenting path with respect to the matching.

Suppose we have a 2-colored graph, say the colors are red and blue.
Suppose $s$ and $t$ are distinct vertices. Let $e$ be a red edge with endpoints
$s$ and $t$. The problem of finding a closed alternating trail through $e$
can be reduced to the alternating reachability problem as follows: Remove $e$
from the multigraph, add two new vertices $s'$ and $t'$, add two new red edges
between $s'$ and $s$ and between $t'$ and $t$. Now look for a $s' - t'$ alternating trail
in the multigraph by taking $S = \{s', t'\}$. A solution to this problem easily yields a solution to the original problem.

Now we are back to the alternating reachability problem. We begin with a definition.

**Definition 0.23** A subset $A \subseteq (V - S)$ is a Tutte set provided no two vertices of $S$ are in the same component of $G - A$ and we can write $A$ as a disjoint union (denoted $\coprod$)

$$A = \coprod_{c \in C} A(c)$$

(some of the $A(c)$ may possibly be empty) such that statements (a), (b) and (c) below hold. First we make a few definitions related to the disjoint union above. A vertex $u \in A$ is said to have color $c$ if $u \in A(c)$ (there is a unique such $c$). An edge $e$ in $G$ is said to be mismatched if it has one endpoint $u$ in $A$, the other endpoint $v$ in $V - A$, and the color of $e$ is different from the color of $u$, or, both endpoints $u, v$ of $e$ are in $A$ and the color of $e$ is different from the colors of $u$ and $v$. 

The following conditions hold.

(a) Let $H$ be a component of $G - A$ that contains a vertex in $S$. Then there are no mismatched edges with endpoints in $H$.

(b) Let $H$ be a component of $G - A$ that does not contain a vertex in $S$. Then there is at most one mismatched edge with an endpoint in $H$.

(c) There are no mismatched edges with both endpoints in $A$.

We have the following observation.

**Lemma 0.24** Suppose a Tutte set $A$ exists. Let $s \in S$. Let $H$ be a component of $G - A$ containing the vertex $t$. Assume that the vertex $s$ is not in $H$ and that there are no mismatched edges with endpoints in $H$. Then there is no alternating $s - t$ trail. In particular, there is no alternating trail between any two distinct vertices in $S$.

The converse of Lemma 0.24 also holds, i.e., if there is no alternating $s - t$ trail, for all $s, t \in S, s \neq t$ then a Tutte set exists.

**Lemma 0.25** Suppose there are no alternating trails between $s$ and $t$. Then there exists a Tutte set $A$ such that the following hold.

1) $s$ and $t$ belong to two different connected components of $G - A$, say $H_1$ and $H_2$ respectively.
2) There are no mismatched edges with endpoints in $H_1$ or $H_2$.
3) The Tutte set $A$ can be computed in polynomial time.
Also by choosing suitable augmenting trails we need to search for alternating trails only polynomially many times. This fact combined with Lemma 0.25 gives us a polynomial time scheme to find feasible balanced flows or to show that no such flow exists.

We would like to mention the following question for future work.

**Question 0.26** Is there a “Hoffman type” condition which involves only the functions $u$, $\ell$ and some kind of “cut”, for a balanced feasible circulation to exist?

We do not know the answer for this question but we believe it has a “nice” answer.

In the case of directed graphs, circulations can be thought of as fluid flows along the arcs meeting the conservation constraint at every node, i.e., total inflow into a node equals the total outflow from the node. This kind of physical intuition is lacking in the case of integral vectors in the alternating cone. This is basically due to the presence of irreducible closed alternating walks whose characteristic vectors are $\{0,1,2\}$-valued. But an irreducible closed alternating trail $T$ corresponds, in some sense, to a unit flow along $T$. Therefore, a natural object to consider is the cone in $\mathbb{R}^E$ generated by the characteristic vectors of closed alternating trails. If we ignore the colors, a closed alternating trail can be decomposed into a sum of cycles in the underlying graph and thus we expect the cone of closed alternating trails to be the intersection of the alternating cone with the cone of cycles of the underlying graph.

Seymour gave a description of the cone of cycles. He proved the following theorem.

**Theorem 0.27** [13] Let $G(V, E)$ be a simple graph, and let $C$ be its collection of cycles. Let $u, \ell : E \to \mathbb{Q}^+$ satisfy $u \geq \ell \geq 0$; then the following are equivalent:

1. there exists $\alpha : C \to \mathbb{Q}^+$ such that $u \geq \sum_{C \in E} \alpha(C) f_C \geq \ell$;
2. for each cut $B$ and $e \in B$, $\ell(e) \leq \sum_{e' \in B - \{e\}} u(e')$.

Here $f_C$ denotes the characteristic vector of the cycle $C$.

In this thesis we prove the following theorem and make a related conjecture.

**Theorem 0.28** A rational vector $c = \{c(e) : e \in \mathbb{R}^E\}$ can be expressed as a sum of alternating trails iff for every cut $B$ of $G(V, E)$ and every edge $e \in B$

$$c(e) \leq \sum_{d \in B - \{e\}} c(d),$$

and “balance condition” holds at every vertex. In other words, the cone of alternating trails is the intersection of the cycle cone of the underlying graph with the alternating cone.
**Conjecture 0.29** Let $G(V,E)$ be a two colored multigraph with $\mathcal{C} = \{R,B\}$ and let $\mathcal{C}$ and $\mathcal{T}$ be its collection of cycles and collection of closed alternating trails respectively. Let $p : E \to \mathbb{Z}^+$ be a balanced map; then the following are equivalent:

1. there exists $\alpha : \mathcal{C} \to \mathbb{Z}^+$ such that $\sum_{C \in e} \alpha(C)f_C = p$;
2. there exists $\alpha : \mathcal{T} \to \mathbb{Z}^+$ such that $\sum_{T \in \mathcal{T}} \alpha(T)f_T = p$. 
Bibliography


