Curriculum Vitae

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Publications

[BGS] A. Bhattacharya, S. K. Ghosh and S. Sarkar, Exploring an unknown polygonal environment with bounded visibility, *Lecture Notes in Computer Science*, No. 2073, 640-648, Springer Verlag, 2001.

[BV] A. Bhattacharya and G. R. Vijayakumar, Effect of Edge-subdivision on Vertex-domination in a Graph, *Discussiones Mathematicae Graph Theory*, 22 (2002) 335-347.

[B] A. Bhattacharya, On a conjecture of Manickam and Singhi, *Discrete Mathematics*, 272 (2003) 259-261.

[Thesis] Ph. D. Thesis, Some problems in combinatorics. (100 pages)

[BV] A. Bhattacharya and G. R. Vijayakumar, Domination in digraphs and variants of domination in graphs, *J. Combin. Inform. System Sci.*, 30 (2005), no. 1-4, 19–24..

[BSS] A. Bhattacharya, S. Sivasubramanian and Murali K. Srinivasan, The polytope of degree partitions, *The Electronic Journal of Combinatorics*, Volume 13(1), 2006, 18 pages.

[Notes] "Quantum Computation, Quantum Error Correcting Codes and Information Theory". I am the scribe of these notes. The lectures were given by Professor K. R. Parthasarathy (Indian Statistical Institute, Delhi). They have been published by Tata Institute of Fundamental Research and some of the copies have been distributed by AMS. (120 Pages.)

[BPS] A. Bhattacharya, U. N. Peled and M. K. Srinivasan, Cones of closed alternating walks and trails, *Linear Algebra and Its Applications*, Volume 423, Issues 2-3, 1 June 2007, 351–365.

[BV] A. Bhattacharya and G. R. Vijayakumar, An integrality theorem of root systems, *European Journal of Combinatorics*, Volume 28, Issue 6, August 2007, 1854–1862.

[BFP] A. Bhattacharya, U. N. Peled and S. Friedland, The Polytope of Dual Degree Partitions, *Linear Algebra and Its Applications*, Volume 426, Issues 2-3, 15 October 2007, 458–461.

[BFP] A. Bhattacharya, U. N. Peled and S. Friedland, On the First Eigenvalue of Bipartite Graphs, *The Electronic Journal of Combinatorics*, Vol. 15, 2008, 23 pages.

[BPS1] A. Bhattacharya, U. N. Peled and M. K. Srinivasan, The cone of balanced subgraphs, *Linear Algebra and its Applications*, Volume 431, Issues 12, 1 July 2009, Pages 266-273 Amitava Bhattacharya, Uri N. Peled, Murali K. Srinivasan

[BS] A. Bhattacharya and N. M. Singhi, Some Approaches to solve General (t, k) design existence problem and other related problems, *Discrete Applied Mathematics*, Volume 161, Issue 9, June 2013, Pages 11801186

[BDMT] A. Bhattacharya, B. DasGupta D. Mubayi and G. Turàn, On optimizing Horn Formula, ICALP'10 Proceedings of the 37th international colloquium conference on Automata, languages and programming Pages 438-450 Springer-Verlag Berlin, Heidelberg.

Submitted papers

[BS] A. Bhattacharya and N. M. Singhi, Characterization of line graphs of r-uniform hypergraphs with bounded pair degree.

[BPS2] A. Bhattacharya, U. N. Peled and M. K. Srinivasan, Alternating Reachability.

Statement:

I am interested to study basic questions in convex polytopes (enumerative and algorithmic aspects), graph theory and algorithms. I am also interested in algebraic combinatorics.

1 Research proposal and planned results

1.1 Degree sequences of r-uniform hypergraphs

The problem to characterize the degree sequence of r-uniform hypergraphs is one of the most basic unsolved problems in the theory of hypergraphs [3]. It is the simplest case of the following basic question. Let n, t, k be positive integers such that $t < k < \frac{n}{2}$. Let $f : \binom{[n]}{k} \to \mathbb{Z}_{\geq 0}$. Let $\partial_t f : \binom{[n]}{t} \to \mathbb{Z}_{\geq 0}$ be defined by

$$\partial_t f(T) = \sum_{T \subset B, B \in \binom{[n]}{k}} f(B)$$

for all sets T in $\binom{[n]}{t}$.

Problem 1.1 *(The general (t,k)-existence problem [31]) Characterize all nonnegative functions $g : \binom{[n]}{t} \to \mathbb{Z}_{\geq 0}$ such that there exists a function $f : \binom{[n]}{k} \to \{0,1\}$ satisfying $\partial_t f = g$.*

Some special cases such as t-designs (g constant), partial Steiner systems (g constant and equal to 1), degree sequences of r-uniform hypergraphs ($t = 1$) and some related problems such as f-vectors of pure simplicial complexes [20, 22, 17] have received much attention during the last three decades. Yet we know very little about the problem. We do not even know whether or not it is NP-complete.

Problem 1.1 is of great interest to statisticians since it includes the existence of designs as a special case. Statisticians are interested to know if there exists designs with some specific parameters. It would be of immense practical and theoretical value if one could find an algorithm (preferably polynomial-time) to solve Problem 1.1.

In [31] a simple necessary and sufficient condition is given which solves Problem 1.1 if f takes values in $\mathbb{Z}$. That paper also contains references to earlier work in this direction.

In general, Problem 1.1 may not be solvable in polynomial time; in [10] it has been shown that some problems related to Problem 1.1 are NP-complete; but for many classes of designs it may be possible to obtain a polynomial-time algorithm.

If we are allowed to repeat edges then the degree sequence problem is easily solvable in polynomial time and there are good characterizations [16, 29]. For graphs this problem is well-studied and there are many elegant characterizations. One of the well-known characterizations is due to Erdős-Gallai [13]. The book [24] gives 9 characterizations. For most of these characterizations a class of graphs called threshold graphs satisfy the characterizations in an extremal way. A graph is called *threshold* if it can be constructed from the one-vertex graph by repeatedly adding either an isolated vertex (i.e., a vertex non-adjacent to all previous vertices) or a dominating vertex (i.e., a vertex adjacent to all previous vertices). For graphs a polytope associated with the degree sequence problem has been studied [21, 26, 33] and it yields a good characterization of degree sequence of graphs. The polytope $DS(n)$ is defined as the convex hull of all possible degree sequences on n vertices. A sequence of nonnegative integers is a degree sequence if and only if it lies in $DS(n)$ and its sum is even. The extreme points of $DS(n)$ are the degree sequences of threshold graphs and vice versa.

We attempt to generalize this approach for hypergraphs. This motivates us to define $DS_r(n)$ as the convex hull of the degree sequences of all r-uniform hypergraphs on n vertices. This polytope has been studied in [4]. That paper showed that the extreme points of $DS_r(n)$ are the degree sequences of the r-threshold hypergraphs, and that the membership and separation problems for $DS_r(n)$ can be solved in polynomial time. We can test whether a nonnegative integer sequence is the degree sequence of some r-threshold hypergraph, in polynomial time using the ellipsoidal method. We do not know any other polynomial-time characterizations. It seems that it is not sufficient for a nonnegative integer sequence to lie in $DS_r(n)$ and satisfy some parity conditions in order for it to be the degree sequence of some r-uniform hypergraphs on n vertices. The facets of the polytope $DS(n)$ have been computed [26] and they can be viewed as a "polytope" version of the

Erdős-Gallai inequalities. In contrast, the facets of $DS_r(n)$ are extremely complicated: it has facet-defining inequalities with Fibonacci numbers as coefficients. So it seems that working with $DS_r(n)$ may be hard.

To proceed we need a few definitions. A subset I of a poset P is called an *ideal* if $q \in I$, $p \in P$ and $p \leq q$ imply $p \in I$. We write the elements of the set $S_r(n) = \binom{[n]}{r}$ as increasing sequences $(a_1 < a_2 < \cdots < a_r)$, $a_i \in [n]$ and we partially order $S_r(n)$ as follows: for $A = (a_1, a_2, \ldots, a_r)$ and $B = (b_1, b_2, \ldots, b_r)$ in $S_r(n)$, $A \leq B$ means $a_i \leq b_i$ for all $i = 1, 2, \ldots, r$. We use "ideal" also to denote an ideal in $S_r(n)$.

It is not hard to observe that a nonnegative integer sequence is the degree sequence of some r-uniform hypergraph on n vertices if and only if it is majorized by the degree sequence of some ideal. This motivates us to consider the problem to characterize the degree sequences of ideals. Some work in this regard has been done in [28].

These observations motivated us to consider the order polytope $O(P)$ of a poset P [32], defined by

$$\begin{aligned} x_p &\geq x_q, && p < q, \;\; p, q \in P, \\ 0 \leq x_p &\leq 1, && p \in P. \end{aligned}$$

Since the constraint matrix of this system is totally unimodular, the extreme points of $O(P)$ are precisely the characteristic vectors of the ideals of P. To optimize linear functions over $O(P)$ we do not need linear programming algorithms such as ellipsoidal methods or interior point methods; network flows can be used [27].

We consider the order polytope $O(S_r(n))$. If x is an extreme point of $O(S_r(n))$, then x is the characteristic vector of an ideal I of $S_r(n)$, and by its definition the degree sequence of I is non-increasing. This degree sequence is $M(n)x$, where $M(n)$ is the incidence matrix of singletons vs. members of $\binom{[n]}{r}$. Thus $M(n)O(S_r(n))$ is the convex hull of all degree sequences of ideals.

To check whether a given $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of an ideal, we need to check whether the equation $M(n)x = d$ has an integral solution $x \in O(S_r(n))$, or equivalently whether $O(S_r(n)) \cap \{x : M(n)x = d\}$ contains an extreme point of $O(S_r(n))$. Suppose we are given two polytopes P_1 and P_2 such that $P_2 \subset P_1$ and we ask whether P_2 contains an extreme point of P_1. This general problem probably does not have a polynomial-time solution, because it is in the class NP and it is not hard to reduce SAT to it. In contrast, we believe that our specific problem is tractable since the constraint matrix of $O(S_r(n))$ and M have a nice combinatorial structure, which is not the case for the corresponding polytope problem for SAT (or other NP-complete problems). So we believe Fourier-Motzkin elimination or cutting-plane methods combined with combinatorial techniques will lead to a polynomial-time recognition of the degree sequences of ideals.

After the degree sequences of ideals have been characterized, we want to use a similar approach to recognize the degree sequences of r-uniform hypergraphs. Set $P_1 = O(S_r(n))$ and $P_2 = \{x : x \in P_1 \text{ and } d \prec Mx\}$. Then P_2 contains an extreme point of P_1 if and only if $(d_1, d_2, \ldots, d_n)$ is the degree sequence of an r-uniform hypergraph [8].

Before the attempt to study $M(n)O(S_r(n))$, $M(n)O(S_2(n))$ was studied. Consider the following question. Let P be a polytope with integral extreme points in $\mathbb{R}^n$ that is closed under coordinate permutations of its points, i.e., $x \in P$ implies $\pi x \in P$, for all permutations π of $[n]$. For example, $DS(n)$ is such a polytope. Let E denote the set of extreme points of P, and let $E_d \subseteq E$ denote the set of extreme points that have non-increasing coordinates. There are two natural ways to define the *asymmetric part* of P. In terms of lattice points we define the asymmetric part of P as the polytope

$$P_d = \text{conv } \{(x_1, x_2, \ldots, x_n) \in P \cap \mathbb{N}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

In terms of linear inequalities we define the asymmetric part of P as

$$P_l = P \cap \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

It is easily seen that $P_d \subseteq P_l$ and $E_d \subseteq$ set of extreme points of P_d. Equality need not hold in these two inclusions. In [7] it was shown that for $P = DS(n)$, $P_d = P_l = O(S_2(n))$, and in this case we call this polytope the *polytope of degree partitions* and denote it by $DP(n)$. Because of the inequalities $x_1 \geq x_2 \geq \cdots \geq x_n$, most of the inequalities defining $DS(n)$ become redundant and $DP(n)$ has only polynomially-many facets while still retaining all the important properties of $DS(n)$. It was observed in [7] that $DP(n)$ has 2^{n-1} vertices, $2^{n-2}(2n-3)$ edges and $(n^2-3n+12)/2$ facets. In [7] is was guessed that $DP(n)$ has $p_i(n)2^{n-1-i}$ faces of dimension i, where $p_i(n)$ is a polynomial in n. It would be interesting to verify this. It seems that $DS(n)$ is similar to the hypercube in many ways (for the hypercubes we also have $P_d = P_l$, and it has 2^{n-i} facets of dimension i).

One can ask the same question for $DS_r(n)$. Computer experiments suggest that a similar conclusion may also hold for $DS_r(n)$. If this is so then it would be very surprising.

Stanley [33] studied a polytope closely related to $DS(n)$. He counted the faces of each dimension for that polytope. In particular he obtained the number of distinct degree sequences of length n. It would be a good result to do a similar study for $DP(n)$.

1.2 Graph theory

Consider a directed graph. Assign a nonnegative real weight to every arc so that at every vertex, the sum of the weights of the incoming arcs is equal to the sum of the weights of the outgoing arcs. The set of all such assignments forms a convex polyhedral cone in the arc space, called the *cone of circulations*, which is a basic object of study in network flow theory. For instance, placing integral upper and lower bounds on every arc and asking whether there is an integral vector in the cone of circulations meeting these bounds leads to Hoffman's circulation theorem (see the book [14]). Now consider an undirected analog of the situation above. Take a graph whose edges have been colored red and blue. Assign a nonnegative real weight to every edge so that at every vertex, the sum of the weights of the incident red edges equals the sum of the weights of the incident blue edges. The set of all such assignments forms a convex polyhedral cone in the edge space, called the *alternating cone*. The basic theory of the alternating cone was studied in [5, 6]. Given a simple graph $G(V, E)$, consider the complete graph on V and color the edges of E red and the other edges blue. Then the dimension of the alternating cone of this 2-colored complete graph is 0 if and only if G is a threshold graph, and this dimension can be used as a measure of non-thresholdness of G. The extreme rays and the integral vectors of the alternating cone of any 2-colored graph (not necessarily complete) were characterized in [5]. The problem of finding an integral vector in the alternating cone satisfying given upper and lower bounds were considered.

This can be viewed as a 2-colored analog of Hoffman's circulation theorem. The key ingredient in the proof of Hoffman's circulation theorem is the *directed reachability problem*: is there a directed s-t path for given vertices s, t, and characterizing the answer in terms of s-t cuts. In our case the problem was reduced [5] to the *alternating reachability problem*: is there an alternating s-t trail for given vertices s, t in a 2-colored graph (a trail can repeat vertices but not edges, and alternating means that the colors of the successive edges in the trail alternate between red and blue). This problem generalizes the problem of searching for an augmenting path with respect to a given a matching in a non-bipartite graph, and was solved in [6] by generalizing the blossom forest algorithm of Edmonds [12]. A non-algorithmic version of this problem was also studied by Tutte [35, 36].

One of the ways to state Hoffman's circulation theorem is the following.

Theorem 1.2 *(Hoffman's circulation theorem [18]) Let $G = (V, E)$ be a directed graph and let $\mathcal{C}$ be its collection of directed cycles. Let $u, \ell : E \to \mathbb{Q}^+$ satisfy $u \geq \ell \geq 0$. Then the following are equivalent:*

1. *there exists $\alpha : \mathcal{C} \to \mathbb{Q}^+$ such that $u \geq \sum_{C \in \mathcal{C}} \alpha(C) f_C \geq \ell$;*
2. *for each $X \subset V$, $u(\delta^+(X)) \geq \ell(\delta^+(V - X))$.*

Here f_C denotes the characteristic vector of C and $\delta^+(X)$ denotes the set of edges with one endpoint in X and directed out of X. Seymour [30] gave an analog of Hoffman's circulation theorem for an undirected graph $G = (V, E)$. By a cut B we mean the set of edges between S and $V - S$ for some $S \subset V$.

Theorem 1.3 *(Seymour [30]) Let $G(V, E)$ be a simple graph, and let $\mathcal{C}$ be its collection of cycles. Let $u, \ell : E \to \mathbb{Q}^+$ satisfy $u \geq \ell \geq 0$. Then the following are equivalent:*

1. *there exists $\alpha : \mathcal{C} \to \mathbb{Q}^+$ such that $u \geq \sum_{C \in \mathcal{C}} \alpha(C) f_C \geq \ell$;*
2. *for each cut B and each $e \in B$ we have $\ell(e) \leq \sum_{e' \in B - \{e\}} u(e')$.*

Essentially, Seymour obtained the inequalities defining the cone generated by the characteristic vectors of cycles, called the *cycle cone*. In the context of a 2-colored graph, it is interesting to ask the analogous question to characterize the cone generated by the characteristic vectors of the closed alternating trails, called the *cone of closed alternating trails*. Clearly the cone of closed alternating trails is contained in the intersection of the alternating cone and the cycle cone. In [6], it was shown that the inclusion holds with equality. In particular the following theorem was proved. We use the notation $E_R(v)$ to denote the set of edges incident on v colored red and $E_B(v)$ to denote the set of edges incident on v colored blue.

Theorem 1.4 *([6]) Let $G(V, E)$ be a 2-colored multigraph and let $\mathcal{T}$ be its collection of closed alternating trails. Let $p : E \to \mathbb{Q}^+$. Then the following are equivalent:*

1. *there exists $\alpha : \mathcal{T} \to \mathbb{Q}^+$ such that $\sum_{T \in \mathcal{T}} \alpha(T) f_T = p$;*
2. *for each cut B and each $e \in B$, we have $p(e) \leq p(B - \{e\})$ (the cut condition), and at every vertex v, $\sum_{f \in E_R(v)} p(f) = \sum_{f \in E_B(v)} p(f)$ (the balance condition).*

Seymour's proof of Theorem 1.3 depends on a "tricky lemma" by Giles and Seymour. The proof of Theorem 1.4 depends on Theorem 1.3 and on the following theorem [6]. Given a 2-colored bridgeless graph G such that at every vertex of G there is at least one incident blue and at least one incident red edge, then G has a closed alternating trail. The proof of this theorem uses the solution of the alternating reachability problem. This theorem also implies the "tricky lemma" of Giles and Seymour (see [6]). In fact, the proof in [6] is modeled after the proof in [30].

Seymour also asked the question when can an integer vector in the cycle cone be written as a sum of characteristic vectors of cycles with repetitions allowed. He conjectured that it is always possible to do so if all the entries in the vector is even. A special case of this is the famous cycle double cover conjecture [19]. Seymour [30] proved a stronger version of his conjecture for planar graphs. His proof uses a version of the 4-color theorem [34].

We can ask a similar question in case of the cone of closed alternating trails. When can an integral vector in the cone of closed alternating trails be written as an sum of closed alternating trails with repetitions allowed? In [6] it is conjectured that this can be done if and only if the vector can be written as a sum of cycles with repetitions allowed in the underlying uncolored graph, and it satisfies the balance condition.

We have a proof of this conjecture for planar graphs. This proof is nontrivial and uses the ideas of Seymour and Karzanov. We believe ideas used by Alspach, Goddyn and Zhang ([1]) we can extend the proof to all graphs which does not include Petersen graph as a minor. In particular we would like to use their idea of "circuit cover-splicing".

Other than this question we are also working at some questions related to fullerene graphs.

Ramsey Theory

One of the well known results in Ramsey theory is the following.

Theorem 1.5 (Van der Waerden) *For any given positive integers c and ℓ, there is some number N such that if the integers $[N]$ ($\{1, 2, ..., N\}$) are coloured with at most c colours, then in $[N]$ there is a monochromatic arithmetic progression of length at least ℓ.*

We would like to consider its generalization to some graded posets, in particular Binomial Posets $\mathbb{B}_n$. By binomial poset $\mathbb{B}_n$ we mean set of all subsets of $[n]$ with the partial order given by set inclusion.

It is easy to show that for any positive integer n there exists an N such that if $\mathbb{B}_N$ is coloured wih two colours then it contains a monochromatic subposet, isomorphic to $\mathbb{B}_n$. We make the following conjecture.

Conjecture 1.6 *For any positive integer n there exists N such that if $\mathbb{B}_N$ is coloured with two colours then there exists an injective map $\sigma : \mathbb{B}_n \to \mathbb{B}_N$, such that*

1. *if $a \leq b$ in $\mathbb{B}_n$ then $\sigma(a) \leq \sigma(b)$ in $\mathbb{B}_N$.*
2. *$\sigma(\mathbb{B}_n)$ is monochromatic,*
3. *if b "covers" a in $\mathbb{B}_n$, ($rank(b) = rank(a) + 1$) then $rank(\sigma(b)) = rank(\sigma(a)) + k$ (k does not depend on a and b).*

We plan to use the techniques used to prove the polynomial van der Waerden theorem. The proof of this conjecture even for $n = 2$ is not trivial.

We believe a similar conjecture might also be true for other posets like "set partitions" (Π_n).

Most of the research problems mentioned here is a part of a joint ongoing project with my collaborators Shmuel Friedland (University of Illinois at Chicago), Navin Singhi (TIFR, India) and Murali Srinivasan (IIT Bombay, India).

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