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Some approaches for solving the general (t, k)-design existence problem and other related problems

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ABSTRACT

In this short survey, some approaches that can be used to solve the generalized (t, k)-design problem are considered. Special cases of the generalized (t, k)-design problem include well-known conjectures for t-designs, degree sequences of graphs and hypergraphs, and partial Steiner systems. Also described are some related problems such as the characterization of f-vectors of pure simplicial complexes, which are well known but little understood. Some suggestions how enumerative and polyhedral techniques may help are also described.

1. Introduction

Arrangements of objects into a collection of sets with specified incidence properties are of great importance in combinatorics and statistics. A special class of configurations is called *designs*. One of the most important types of design is balanced incomplete block design (BIBD).

Definition 1.1. Let X be a set of size v. A BIBD is an arrangement of elements of X into b sets (blocks) such that each set contains exactly k elements, each element is contained in exactly r different sets, and every $C \in \binom{X}{2}$ is contained in exactly λ sets.

BIBDs have many applications in the design of experiments in statistics. Hence much effort has been spent in developing methods for constructing BIBDs. There are many methods of constructions known. Most of the techniques for constructing BIBDs use results from finite groups, finite fields, and quadratic forms. Also many interesting composition techniques have been developed for the construction of BIBDs from smaller BIBDs. Bose was one of the first to give methodical constructions using groups and finite fields. Hanani solved the existence problem for BIBDs for all cases, when $k \le 5$. Later Hanani, Bose, Shrikhande, Ray-Chaudhuri, Wilson, and several others improved these methods considerably. Finally, Wilson showed that for a given k and λ a BIBD exists, when v is sufficiently large. For more details, the reader is referred to [2,9]. But the general question, namely for what parameters a BIBD exists, has not yet been solved.

A natural generalization of a BIBD is a t-(v, k, λ) design.

Definition 1.2. Let X be a set of size v. A t-(v, k, λ) design is a collection of subsets of X of size k such that any element of $\binom{X}{t}$ is contained in exactly λ elements of this collection.

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The theory of t-designs is a well-developed field, and there are many interesting conjectures about these designs (see [2,7,32,33,23]).

Let $\mathcal{C}$ be a collection of subsets of X such that it is a t-(v, k, λ) design. Suppose that we want to count all possible pairs $((S, T), C)$ where S is any fixed subset of X of size i, T is a subset of X of size t, C is an element of $\mathcal{C}$, and $S \subseteq T \subseteq C$. We can do this count in two different ways. First we can count all t-subsets of X that contain S then count the number of times each t-subset appears in the collection $\mathcal{C}$. This implies the number of such pairs to be $\lambda \binom{v-i}{t-i}$. Another way would be counting all the t-subsets of C that contain S and then counting all the elements C of $\mathcal{C}$ that contain S. This implies the number to be $I \binom{k-i}{t-i}$, where I is an integer. Since these two numbers are equal, we have

$$\lambda \binom{v-i}{t-i} \equiv 0 \bmod \binom{k-i}{t-i}, \quad \text{for all } 0 \le i \le t.$$

Thus every t-(v, k, λ) design must satisfy the above equations.

A central problem in the theory of t-designs is the following.

Conjecture 1.3 (*Existence Conjecture*)**.** *Given $0 \le t < k \le v$, with v sufficiently large compared to k, a t-(v, k, λ) design exists if and only if*

$$\lambda \binom{v-i}{t-i} \equiv 0 \bmod \binom{k-i}{t-i}, \quad \textit{for all } 0 \le i \le t.$$

So far, the conjecture has been proved to be true for $t = 1$ and $t = 2$ only. In fact, most of the known results for t-designs are only for the case $t = 2$ (see [2]). For the case $t = 2$, as was earlier remarked, the conjecture in the above general form was proved by Wilson [33].

We now need to introduce some notation.

We will denote by $\mathbb{P}(X)$ the set all subsets of X, and by $\mathbb{P}_k(X)$ the set of all k-subsets of X, $0 \le k \le v$. We will denote by $V_k(X)$ the set of all rational-valued functions $f : \mathbb{P}_k(X) \to \mathbb{Q}$. Observe that $V_k(X)$ is a vector space over $\mathbb{Q}$, of dimension $\binom{v}{k}$. The set $M_k(X) \subseteq V_k(X)$ of all integral-valued functions is a module of rank $\binom{v}{k}$ over the ring of integers $\mathbb{Z}$.

Now, let $0 \le t \le k \le v$. For $f \in V_k(X)$, define $\partial_t f \in V_t(X)$ by $\partial_t f(T) = \Sigma f(B)$, where the sum is over all B satisfying $T \subseteq B$.

The function $j_k \in V_k(X)$ is defined by $j_k(B) = 1$ for all $B \in \mathbb{P}_k(X)$.

Let $N_{t,k} = N_{t,k}(X)$ denote the $\binom{v}{t} \times \binom{v}{k}$ matrix defined by

$$N_{t,k}(T, B) = \begin{cases} 1 & \text{if } T \subseteq B, \\ 0 & \text{otherwise,} \end{cases}$$

where $T \in \mathbb{P}_t(X)$ and $B \in \mathbb{P}_k(X)$.

Now, let $D = (X, f)$, where $f \in V_k(X)$. D is called a *rational* t-(v, k, λ) design if $\partial_t f = \lambda j_t$. Such a pair is called a *signed* t-(v, k, λ) design if f is integral, and a t-(v, k, λ) design if f is integral and non-negative ($f \ge 0$), i.e., for all $B \in \mathbb{P}_k(X)$, $f(B) \ge 0$ and $f(B) \in \mathbb{Z}$. The number $f(B)$ may be thought of as the frequency with which the k-subset B occurs in the design. When $f(B)$ is non-zero, the k-subset B is called a block of the design.

For a collection $\mathcal{C}$ of subsets of size k, consider its frequency function $f \in M_k(X)$ to be the function defined by $f(B) =$ the number of times B occurs in the collection $\mathcal{C}$. We note that $\partial_t f = \lambda j_t$ if and only if $\mathcal{C}$ is a t-(v, k, λ) design. Thus, the two definitions of a t-(v, k, λ)-design we have given are essentially the same.

We will fix X throughout this paper. Then we may refer to f itself as a t-design if it satisfies above conditions. For the usual definition of t-designs as a family of subsets, f corresponds to the frequency vector of occurrence of k-subsets in the family.

Another well-known problem concerning families of sets is the following characterization problem for degree sequences of a uniform hypergraph.

Problem 1.4 (*Characterization Problem*)**.** Let $h \in V_t(X)$, $t = 1$. Give necessary and sufficient conditions, which can be verified algorithmically (in polynomial time), for the existence of an $f \in V_k(X)$ such that $f(B) \in \{0, 1\}$ for all $B \in \mathbb{P}_k(X)$ and $\partial_1 f = h$.

Note that, for $k = 2$, the problem corresponds to characterizing the degree sequence of the graphs. The well-known Havel–Hakimi theorem and the Erdős–Gallai theorem are such characterizations. Also, for general k, if we allow multiple edges, that is we replace the condition $f(B) \in \{0, 1\}$ by the condition $f(B)$ is a non-negative integer, then a good characterization is provided by the well-known Gale–Ryser theorem [15,25]. These are essentially the only general cases for which a good characterization is known.

We can now state a general (t, k)-existence problem, special cases of which include many well-known problems in combinatorics, apart from the above problems. Some examples are existence problems on partial Steiner systems and characterization of f-vector of a pure simplicial complex. The problem of characterization of a pure simplicial complex itself includes many well-known interesting problems in the theory of designs (for more details, see [2,26,27] and Exercise 8.16 in [35]). The general (t, k)-existence problem which was first described in [28] is perhaps very hard; so far no good algorithm has been found. However, one can hope to have asymptotic existence results, which may help in settling some of these conjectures.

Problem 1.5 (*(t, k)-Existence Problem*)**.**

(a) Characterize all $h \in V_t(X)$ such that there exists a non-negative integral function $f \in V_k(X)$ with $\partial_t f = h$.
(b) Characterize all $h \in V_t(X)$ such that there exists $f \in V_k(X)$ with $\partial_t f = h$ and $f(B) \in \{0, 1\}$ for all $B \in \mathbb{P}_k(X)$.

In Section 2, we briefly outline how the idea of tags and the principle of inclusion–exclusion can be used to solve the (t, k)-existence problem over integers. We hope that the ideas presented can be used to solve the problem, at least in the asymptotic case.

In Section 3, we consider Problem 1.4, a special case of Problem 1.5(b). We outline a possible way to solve it using polyhedral techniques.

2. Tags and the general (t, k)-existence problem

One important tool for t-designs or the problems above is studying the function $\partial_t : V_k(X) \to V_t(X)$ (or the restriction $\partial_t/M_k(X) : M_k(X) \to M_t(X)$) as a linear transformation.

This method was initially developed by Wilson [32] and Graver and Jurkat [17]. In particular, the latter described an interesting set of generators of the subspace $\ker \partial_t$ of $V_k(X)$. Note that every integral element of $\ker \partial_t$ is in $\ker(\partial_t/M_k(X))$, and can be thought of as a signed t-(v, k, λ) design with $\lambda = 0$. Such designs are called 0-*designs* or a (v, t, k)-*trade* (see [12,11]).

In [28], a very useful description of this set of generators for the $\mathbb{Z}$-module $\ker(\partial_t/M_k)$ (or $\ker \partial_t$ as a subspace of $V_k(X)$ is given. This description is due to Graham et al. [16].

Let $A = \{y_1, y_2, \ldots, y_{2t+2}, w_1, w_2, \ldots, w_{k-t-1}\}$ be a subset of X, which need not be a chain. Define $f_A \in V_k(X)$ as follows. Consider a polynomial P_A with variables from the set $A \subset X$ defined by

$$P_A = (y_1 - y_2)(y_3 - y_4) \cdots (y_{2t+1} - y_{2t+2}) w_1 w_2 \ldots w_{k-t-1}. \tag{1.2}$$

Let $B = \{r_1, r_2, \ldots, r_k\}$, $B \in \mathbb{P}_k(X)$. Now, define $f_A(B)$ to be the coefficient of the monomial $r_1 r_2 \cdots r_k$ in P_A. Thus $f_A(B)$ is ± 1 or 0.

It can be easily seen that $f_A \in \ker \partial_t$, and Graham et al. [16] showed that they generate the above-described kernels; in fact, they also described a basis contained in this set of generators.

For any two chains $B_1 = \{y_1 \geq y_2 \geq \cdots \geq y_k\}$ and $B_2 = \{w_1 \geq w_2 \geq \cdots \geq w_k\}$ of X, we will say that $B_1 < B_2$ in *lexicographic ordering* if there exists a j, such that $1 \leq j \leq k$, $y_i = w_i$ for $1 \leq i < j$ and $y_j < w_j$.

Now we can describe the basic philosophy behind the development of the concept of tags. Suppose we assume in (1.2) that

$$y_{2j} \leq y_{2j-1} \quad 1 \leq j \leq t + 1.$$

Also, suppose that $B_1 = \{y_1, y_3, \ldots, y_{2t+1}, w_1, w_2, \ldots, w_{k-t-1}\}$.

For any function f, define *support* (f) by

$$\text{support } (f) = \{B \mid B \text{ in domain of } f,\ f(B) \neq 0\}.$$

For $f \in V_k(X)$, define ℓ-max$(f) = B$, where B is the maximal element of support (f), in lexicographic ordering.

Now, suppose that $f \in V_k(X)$ with ℓ-max$(f) = B_1$.

Consider $g = f - f(B_1)f_A$. Clearly, $g \in V_k(X)$. It can be easily seen that

$$\partial_t g = \partial_t f - \partial_t f(B_1) f_A = \partial_t f, \quad \text{and} \quad \ell\text{-max}(g) < \ell\text{-max}(f).$$

Thus we can think of the process of obtaining g from f as pushing ℓ-max down in lexicographic ordering without disturbing the ∂_t values. This suggests that, in order to study the properties of ∂_t, there should be a natural set of $\binom{v}{t}$ elements of $\mathbb{P}_k(X)$, smallest in lexicographic ordering, under the above pushing process, which generates an image of ∂_t.

We also note that some kind of pushing in lexicographic ordering is a standard technique and has been applied to prove strong basic results. Examples are the Kruskal–Katona theorem on f-vectors of simplicial complex, the upperbound theorem on convex polytopes, characterizations of Hilbert functions of graded algebras, and the Erdős–Ko–Rado theorem (see [29,7,2,26]).

It is this idea of pushing in the context of ∂_t that leads to the definition of a tag and gives in a natural manner the decomposition of $\mathbb{P}_k(V)$ into subsets of size $\binom{v}{\ell} - \binom{v}{\ell - 1}$, $0 \leq \ell \leq k$. This concept was first developed in [8]. Concepts related to tags were also considered in [1]. In [8], the definition of an association of a tag with a k-subset was given using reflection and complementation maps. It was also shown in [8] that this concept simplifies many well-known results on t-designs and other related problems. The above-mentioned decomposition of $\mathbb{P}_k(X)$ into sets of size $\binom{v}{\ell} - \binom{v}{\ell - 1}$ corresponds to the well-known decomposition of vector space $V_k(X)$ (or module $M_k(X)$) using null spaces $N_{t,k}$, $0 \le t < k$ (see [34]).

In [28], a much simpler definition of association of a tag with a k-subset in the natural lexicographic setting is given. In [28], the concept of a tagdual was also introduced. Tag and tagdual play crucial roles in studying the functions ∂_t and $\partial_t \mid M_k$. Though this new definition of a tag associated with a k-subset is slightly different from that in [8], it was shown in [28] that all results in [8] are valid for this new description with the necessary modifications.

We now make some formal definitions and then state one of the important theorems proved in [28].

Definition 2.1. Let $W \subseteq X$, $W = \{w_1, \dots, w_\ell\}$, be a chain. We will call w_i the *ith element of* W. The set W will be called an ℓ*-tag* or simply a *tag* if $v \ge 2\ell$ and $w_i \ge x_{2i}$ for all $i \in [\ell]$.

For $\ell = 0$, a 0-tag is defined to be the empty set ϕ.

Definition 2.2. The dual of an ℓ-tag W is defined to be the set $W' = \{w_1, w_2, \dots, w_\ell\}$ such that the following hold.

(a) $W \cap W' = \emptyset$.
(b) $w_i' < w_i$ for all i in $[\ell]$.
(c) W' is the largest ℓ-set satisfying (a) and (b).

Now, let W be an ℓ-tag. For each $x \in X$, define $\rho(x, W)$ and $\sigma(x, W)$ by

$$\rho(x, W) = \{w \mid w \in W', w \le x\}$$

and

$$\sigma(x, W) = \{w \mid w \in W,\ x < w\}.$$

For any $x \in X$, let $I(x)$ denote the set of all elements in X less than or equal to x. Let W be an ℓ-tag and W' its dual; then, by $y(k, W)$ (or $y(W)$ when there is no confusion), we mean the unique element $x \in X - (W \cup W')$ which satisfies $|I(x) - (W \cup W')| = k - \ell$.

When x is equal to $y(W)$, we denote $\rho(x, W)$ and $\sigma(x, W)$ by $\rho(W)$ and $\sigma(W)$, respectively.

Now we can define the relation |.

Definition 2.3. Let $B \in \mathbb{P}_k(X)$ and $W \in \mathrm{Tag}_k(X)$. We will say that W *divides* B or $W \mid B$ if $\rho(W) \cap B = \emptyset$ and $\sigma(W) \subseteq B$.

Now we define the concept of Tags on a function.

Definition 2.4. Let $V(\mathrm{Tag}_m(X))$ be the vector space of all functions $g : \mathrm{Tag}_m(X) \to Q$, $0 \le m \le [v/2]$, and $M(\mathrm{Tag}_m(X))$ be the set of integral elements of $V(\mathrm{Tag}_m(X))$.

We define a linear operator $\mathrm{Tag}_t : V_k(X) \to V(\mathrm{Tag}_t(X))$ as follows.

For $f \in V_k(X)$, define $\mathrm{Tag}_t f$ by

$$\mathrm{Tag}_t f(W) = \sum f(B),$$

where the sum is over all $B \in \mathbb{P}_k(X)$ satisfying $W \mid B$.

If $t = k$, $\mathrm{Tag}_k f$ will be denoted by $\mathrm{Tag}\, f$.

Remark 2.5. Tag_t is clearly a linear transformation, and results in [28] and [8] showed that Tag_t plays a very similar role in studying t-designs and other combinatorial objects as ∂_t.

For $B \in \mathbb{P}_k(X)$, define $\delta_B : \mathbb{P}_k(X) \to \mathbb{Z}$ by

$$\delta_B(B_1) = \begin{cases} 1 & \text{if } B_1 = B, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for $W \in \mathrm{Tag}_t(X)$, let us define by δ_W

$$\delta_W(W_1) = \begin{cases} 1 & \text{if } W_1 = W, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{\delta_W \mid W \in \mathrm{Tag}_t(X)\}$ clearly forms a basis for $M(\mathrm{Tag}_t(X)\})$ as a $\mathbb{Z}$-module or $V(\mathrm{T\,ag}_t(X))$ over Q. Similarly, $\{\delta_B \mid B \in \mathbb{P}_k(X)\}$ forms a basis for $M_k(X)$ or $V_k(X)$.

Lemma 2.6. *Let* $W \in \mathrm{Tag}_t(X)$*. There exists a function* $\mu_W \in M_k(X, t) \subseteq V_k(X, t)$ *such that* $\mathrm{Tag}_t \mu_W = \delta_W$.

The following result follows from the above lemma and the inclusion–exclusion principle.

Theorem 2.7. *Let $h : \mathbb{P}_t(X) \to Q, f : \mathbb{P}_k(X) \to Q$, and $\partial_t f = h$. Suppose that $0 \le t \le k \le v, I \cap J = \phi$, and $| I \cup J | \le t$. Then*

$$\sum f(B) = \sum \frac{(-1)^{|J'|}}{\binom{k-|I|-|J'|}{t-|I|-|J'|}} \left(\sum h(T)\right),$$

where the sum on the left-hand side is over all $B \in \mathbb{P}_k(X)$ satisfying $B \cap J = \phi$ and $I \subseteq B$, the first sum on the right-hand side is over all J' satisfying $J' \subseteq J$, and the second sum on the right-hand side is over all $T \supseteq I \cup J'$.

This implies the following result, first proved in this form by Wilson [32]. A similar result was also proved independently by Graver and Jurkat [17].

Corollary 2.8. *Let $t \le k \le v - t$. The necessary and sufficient conditions for the existence of a function $f : \mathbb{P}_k(X) \to \mathbb{Z}$ satisfying $\partial_t f = h$ for a given $h : \mathbb{P}_t(X) \to \mathbb{Z}$ are that, for all $I \subseteq X$, $| I | \le t$,*

$$\sum h(T) = 0 \mod \binom{k- | I |}{t- | I |},$$

where the sum is over all T in $\mathbb{P}_t(X), I \subseteq T$.

This shows that part (a) of the (t, k)-existence problem can be solved if one looks for integral functions instead of non-negative integral functions. But the concept of Tags looks very promising for solving part (a) of the (t, k)-existence problem at least in the asymptotic case. A non-negative integral solution for part (a) of the (t, k)-existence problem essentially asks for the existence of integral points in a convex polytope. In the special case of t-designs this polytope was studied in [6]. Some inequalities corresponding to this polytope are described in [28].

3. Degree sequence of hypergraphs

In this section, we outline a possible way to solve Problem 1.4. Instead of trying to find necessary and sufficient conditions we will like to get an answer of the following form. Given a function h, we would like to construct the function f in polynomial time or report that none exists. Some of the ideas presented in this section are the result of discussions with Friedland, Peled, and Srinivasan.

In general, Problem 1.5 may not be solvable in polynomial time; in [10], it has been shown that some problems related to Problem 1.5 are NP-complete; but for many classes of designs it may be possible to obtain a polynomial-time algorithm. We believe that Problem 1.4 may be solvable in polynomial time.

If we are allowed to repeat edges, then the degree sequence problem is easily solvable in polynomial time, and there are good characterizations [15,25]. For graphs, this problem is well studied, and there are many elegant characterizations. One of the well-known characterizations is due to Erdős and Gallai [14]. The book [20] gives nine characterizations. For most of these characterizations, a class of graphs called threshold graphs satisfies the characterizations in an extremal way. A graph is called a threshold graph if it can be constructed from the one-vertex graph by repeatedly adding either an "isolated" vertex (i.e., a vertex non-adjacent to all previous vertices) or a "dominating" vertex (i.e., a vertex adjacent to all previous vertices). For graphs, a polytope associated with the degree sequence problem has been studied in [19,21,30], and it yields a good characterization of the degree sequence of graphs. The polytope $DS(n)$ is defined as the convex hull of all possible degree sequences on n vertices. A sequence of non-negative integers is a degree sequence if and only if it lies in $DS(n)$ and its sum is even. The extreme points of $DS(n)$ are the degree sequences of threshold graphs, and vice versa.

We attempt to generalize this approach for hypergraphs. This motivates us to define $DS_r(n)$ as the convex hull of the degree sequences of all r-uniform hypergraphs on n vertices. The polytope $DS_r(n)$ has been studied in [3]. It was shown in that paper that the extreme points of $DS_r(n)$ are the degree sequences of the r-threshold hypergraphs, and that the membership and separation problems for $DS_r(n)$ can be solved in polynomial time. We can test whether a non-negative integer sequence is the degree sequence of some r-threshold hypergraph in polynomial time, using the ellipsoidal method. We do not know any other polynomial-time characterizations. It seems that it is not sufficient for a non-negative integer sequence to lie in $DS_r(n)$ and satisfy some parity conditions in order to be the degree sequence of some r-uniform hypergraphs on n vertices. The facets of the polytope $DS(n)$ have been computed [21], and they can be viewed as a "polytope" version of the Erdős–Gallai inequalities. In contrast, the facets of $DS_r(n)$ are extremely complicated: it has facet-defining inequalities with Fibonacci numbers as coefficients. So it seems that working with $DS_r(n)$ may be much harder.

To proceed further we need a few definitions. A subset I of a poset P is called an *ideal* if $q \in I, p \in P$, and $p \le q$ imply that $p \in I$. We consider the elements of the set $S_r(n) = \binom{[n]}{r}$ as increasing sequences $(a_1 < a_2 < \cdots < a_r)$, $a_i \in [n]$, and we partially order $S_r(n)$ as follows. For $A = (a_1, a_2, \ldots, a_r)$ and $B = (b_1, b_2, \ldots, b_r)$ in $S_r(n)$, $A \le B$ means that $a_i \le b_i$ for all $i = 1, 2, \ldots, r$. We use "ideal" also to denote an ideal in the partially ordered set $S_r(n)$.

It is not hard to observe that a non-negative integer sequence is the degree sequence of some r-uniform hypergraph on n vertices if and only if it is majorized by the degree sequence of some ideal. This was first observed in [13]. This motivates us to consider the problem to characterize the degree sequences of ideals. Some work in this regard has been done in [24]. In [18], the degree sequence problem on hypergraphs was also considered. It gives various inclusion relations between different definitions of threshold graphs (see also [3]) and links this problem to representation theory.

These observations motivated us to consider the order polytope $O(P)$ of a poset P [31], defined by the inequalities

$$x_p \geq x_q, \quad p < q,\ p, q \in P,$$
$$0 \leq x_p \leq 1, \qquad p \in P.$$

Since the constraint matrix of this system is totally unimodular, the extreme points of $O(P)$ are precisely the characteristic vectors of the ideals of P. To optimize linear functions over $O(P)$, we do not need linear programming algorithms such as ellipsoidal methods or interior point methods. The simple method of optimizing flows in a network can be used [22].

We consider the order polytope $O(S_r(n))$. If x is an extreme point of $O(S_r(n))$, then x is the characteristic vector of an ideal I of $S_r(n)$. Also, the degree sequence of an ideal I is non-increasing. This degree sequence is $M(n)x$, where $M(n)$ is the incidence matrix where the rows are indexed by singletons and the columns are indexed by members of $\binom{[n]}{r}$. Thus $M(n)O(S_r(n))$ is the convex hull of all degree sequences of ideals.

To check whether a given $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of an ideal, we need to check whether the equation $M(n)x = d$ has an integral solution $x \in O(S_r(n))$, or equivalently whether $O(S_r(n)) \cap \{x : M(n)x = d\}$ contains an extreme point of $O(S_r(n))$. Suppose that we are given two polytopes P_1 and P_2 such that $P_2 \subset P_1$, and we ask whether P_2 contains an extreme point of P_1. This general problem probably does not have a polynomial-time solution, because it is in the class NP and it is not hard to reduce SAT (satisfiability problem) to it. In contrast, we believe that our specific problem is tractable, since the constraint matrix of $O(S_r(n))$ and M have nice combinatorial structures, which is not the case for the corresponding polytope problem for SAT (or other NP-complete problems). So we believe Fourier–Motzkin elimination or cutting-plane methods combined with combinatorial techniques will lead to a polynomial-time recognition algorithm for the degree sequences of ideals.

After the degree sequences of ideals have been characterized, we want to use a similar approach to recognize the degree sequences of r-uniform hypergraphs. Set $P_1 = O(S_r(n))$ and $P_2 = \{x : x \in P_1 \text{ and } d \prec Mx\}$. Then P_2 contains an extreme point of P_1 if and only if $(d_1, d_2, \ldots, d_n)$ is the degree sequence of an r-uniform hypergraph [4].

Before the attempt to study $M(n)O(S_r(n))$, $M(n)O(S_2(n))$ was studied. Consider the following question. Let P be a polytope with integral extreme points in $\mathbb{R}^n$ that is closed under coordinate permutations of its points, i.e., $x \in P$ implies that $\pi x \in P$, for all permutations π of $[n]$. $DS(n)$ is an example of such a polytope. Let E denote the set of extreme points of P, and let $E_d \subseteq E$ denote the set of extreme points that have non-increasing coordinates. There are two natural ways to define the *asymmetric part* of P. In terms of lattice points, we define the asymmetric part of P as the polytope

$$P_d = \text{conv}\{(x_1, x_2, \ldots, x_n) \in P \cap \mathcal{N}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

In terms of linear inequalities, we define the asymmetric part of P as

$$P_l = P \cap \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

It is easily seen that $P_d \subseteq P_l$ and $E_d \subseteq$ set of extreme points of P_d. Equality need not hold in these two inclusions. In [5], it was shown that, for $P = DS(n)$, $P_d = P_l = O(S_2(n))$, and in this case we call this polytope the *polytope of degree partitions* and denote it by $DP(n)$. Because of the inequalities $x_1 \geq x_2 \geq \cdots \geq x_n$, most of the inequalities defining $DS(n)$ become redundant, and $DP(n)$ has only polynomially many facets while still retaining all the important properties of $DS(n)$. It was observed in [5] that $DP(n)$ has 2^{n-1} vertices, $2^{n-2}(2n-3)$ edges, and $(n^2 - 3n + 12)/2$ facets. In [5], is was guessed that $DP(n)$ has $p_i(n)2^{n-1-i}$ faces of dimension i, where $p_i(n)$ is a polynomial in n. It would be interesting to verify this. It seems that $DS(n)$ is similar to the hypercube in many ways (for the hypercubes we also have $P_d = P_l$, and it has 2^{n-i} facets of dimension i).

One can ask the same question for $DS_r(n)$. Computer experiments suggest that a similar conclusion may also hold for $DS_r(n)$. It would be very surprising if this is the case.

Stanley [30] studied a polytope closely related to $DS(n)$. He counted the faces of each dimension for that polytope. In particular, he obtained the number of distinct degree sequences of length n. It would be a good result to do a similar study for $DP(n)$.

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