Determination of cusp forms on $GL(2)$ by coefficients restricted to quadratic subfields
(with an appendix by Dipendra Prasad and Dinakar Ramakrishnan)

M. Krishnamurthy

Department of Mathematics, University of Iowa, Iowa City, IA 52242, United States

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ABSTRACT

Given $E/F$ a quadratic extension of number fields and a cuspidal representation $\pi$ of $GL_2(\mathbb{A}_E)$, we give a full description of the fibers of the Asai transfer of $\pi$. We then determine the extent to which Fourier coefficients defined by integral ideals of $F$ determine the representation $\pi$.

1. Introduction

Let us fix $E/F$ a quadratic extension of number fields and let $\Gamma_F$ denote the absolute Galois group of $F$. For any number field $k$, we write $\mathbb{A}_k$ to denote its ring of adèles, and $C_k$ to denote its idèle class group. Consider the representation

$$r: GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rtimes \Gamma_F \to GL_4(\mathbb{C}) \times \Gamma_F$$

given by

E-mail address: mkrishna@math.uiowa.edu.

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This is a special case of a general construction known as tensor induction. (See [4, §13].) We call $r$ the Asai representation (with respect to $E/F$) in this paper. Since $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rtimes \Gamma_F$ is the Langlands dual group of the restriction of scalars $R_{E/F}GL_2$, we can view $r$ as a homomorphism of $L$-groups. The corresponding global functoriality is established by the author [12] and Dinakar Ramakrishnan [17] using two different approaches. It should be noted that, although we used a different theory of $L$-functions in [12], our proof there still relies on Dinakar Ramakrishnan’s descent criterion. Now, given a cuspidal representation $\pi$ of $GL_2(\mathbb{A}_F)$, let $As(\pi)$ denote this functorial transfer, called the Asai transfer of $\pi$ – it is an isobaric automorphic representation of $GL_4(\mathbb{A}_F)$. We refer the reader to Section 2 for more details.

In [12], we described the fibers of the Asai transfer when $\pi$ is not dihedral, i.e., not automorphically induced from a character. In this paper, we address the dihedral case, thereby completing the description of the fibers of the Asai transfer. Then, as a consequence, we show that it is essentially determined by its Whittaker–Fourier coefficients defined through the subgroup $GL_2(\mathbb{A}_F)$. To be precise, let $\theta$ denote the non-trivial automorphism of $E/F$. The main result (cf. Theorem 4.0.2, Theorem 4.0.3) we prove here is thus: If $\pi$ and $\pi'$ are two cuspidal representations of $GL_2(\mathbb{A}_E)$ such that their central characters agree on $C_F$ and their Fourier coefficients $\lambda_\pi(m)$ coincide for all integral ideal $m$ of $F$, then either $\theta$ or $\theta' = \theta^{-1}$ is an abelian twist $\pi' \otimes \psi$ of $\pi'$ with $\psi$ having trivial restriction to $C_F$. This is similar to the way the Hecke eigenforms of half integral weight are determined by restriction of the Fourier coefficients indexed by fundamental discriminants [13]. The crucial point is that our hypothesis on $\pi$ and $\pi'$ guarantee that both $\pi$ and $\pi'$ have the same Asai transfer. The problem then becomes one of describing the fibers of the Asai transfer.

This may be of interest to researchers in analytic number theory. In fact the problem was brought to the author’s attention by Dinakar Ramakrishnan based on his conversations with Duke and Kowalski. More importantly, our result is a specific instance of a general hope that a cuspidal automorphic representation (of a reductive group) should be determined by an arithmetically defined subset of its coefficients. We refer the reader to [5,13] for interesting examples along these lines and applications thereof. Very recently, Abhishek Saha has proved that holomorphic cusp forms on $GSp(4)$ are determined by a (naturally defined) fundamental subset of the Fourier–Jacobi coefficients [19] by extending the results of [13] to non-eigenforms. In the context of this paper, the fundamental coefficients defined in terms of the subgroup $GL_2/F$ determine the cuspidal representation $\pi$.

It is also worth mentioning that there are some purely representation theoretic consequences. For example, as explained in [19], the non-vanishing of a fundamental coefficient implies the existence of global Bessel models. In another recent work by U.K. Anandavardhanan and Dipendra Prasad [1] the description of the fibers of the Asai transfer is used in answering certain local-global questions in the theory of distinguished representations. Indeed, in [1, §5], the authors have given an alternate – and better – proof of Theorem 2.0.4. However, the proof here is more in line with our proof of the analogous theorem in the non-dihedral case [12] and may have its own merits. For example, one can read off the various possible isobaric types of $As(\pi)$ from our proof. (This basically amounts to giving the multiplicity of the trivial representation in $As(\pi) \otimes As(\pi)$.)

Let us now elaborate on the methods used. Suppose $\pi$ and $\pi'$ are such that $\pi \otimes \psi \simeq \pi'$ for some idèle class character of $\psi$ of $E$ with $|\psi|_{C_F} = 1$, then it is easy to see that $As(\pi) \simeq As(\pi')$. In [12], we proved the converse, at least when both $\pi$ and $\pi'$ are not dihedral. As mentioned earlier, we address the dihedral case in this paper. Namely, if $\pi$ is a dihedral cuspidal representation of $GL_2(\mathbb{A}_E)$, let $\sigma$ be the corresponding two-dimensional representation of the global Weil group $W_F$. If $\pi'$ is another cuspidal representation of $GL_2(\mathbb{A}_E)$ such that $As(\pi) \simeq As(\sigma')$, then $\pi'$ is also dihedral as was shown in [12, §7]: let $\sigma'$ be the two-dimensional representation of $W_E$ corresponding to $\pi'$. Now, the determination of the fibers of the Asai transfer can be carried out on the “Weil-group” side. Namely, our hypothesis yields that $As(\sigma) \simeq As(\sigma')$. Here $As(\sigma)$ (resp. $As(\sigma')$) is the four-dimensional representation of $W_F$ obtained via tensor induction from $\sigma$ (resp. $\sigma'$) (cf. Section 2). By restricting these representations to $W_E$, we get $\sigma \otimes \sigma^{\theta} \simeq \sigma' \otimes \sigma^{\theta'}$, where $\theta$ is the non-trivial Galois automorphism of $E/F$. It is then a straightforward argument
to see that the pair \((\sigma, \sigma^0)\) is determined uniquely up to a twist by a character \((\chi, \chi^{-1})\). (See Lemma 2.0.1.) Thus one concludes that \(\pi \otimes \chi \simeq \pi'\) or \(\pi^0 \otimes \chi \simeq \pi'\) for some idèle class character \(\chi\) of \(E\).

The rest of the argument is to show that \(\chi\) can in fact be chosen so that \(\chi|_{C_F} = 1\). The idea here is to analyze the poles of \(L^{s}(s, \Pi \times \tilde{\Pi})\) at \(s = 1\), where \(\Pi = As(\pi) \simeq As(\pi')\), through the decomposition of \(\Pi\) into its isobaric constituents. On the one hand, we have \(\text{ord}_{s=1} L^{s}(s, \Pi \times \tilde{\Pi}) = 2, 3,\) or \(4\). On the other hand, we have the identity \(L^{s}(\Pi \times \tilde{\Pi}) = L^{s}(\pi \times \tilde{\pi}', R)\), where \(R\) is the natural analogue of \(r\) defined from the \(L\)-group of \(RE/GL_2\) to that of \(GL_{16}\) via tensor induction. This identity is simply a reflection of the fact that tensor induction commutes with taking tensor products and duals. Now, one can factorize \(L^{s}(\pi \times \tilde{\pi}', R)\) further using a general property of tensor induction, namely its behavior with respect to sums of representations, and analyze the order of the pole at \(s = 1\). Finally, a comparison of the order using the resulting identity of \(L\)-functions yields the desired result. (See the proof of Theorem 2.0.4.) Needless to say, a key ingredient in our analysis is the cuspidality criterion of \(As(\pi)\) [17,18].

The paper is organized as follows. In Section 2, we recall the notion of the Asai transfer, both on the automorphic and on the “Weil-group” side. In the dihedral case, we discuss the properties of the Asai transfer and its fibers in detail. In Section 3, which can be read independent of Section 2, we present two kinds of Dirichlet series that one can attach to a cuspidal representation of \(GL_2(\mathbb{A}_E)\). In Section 3.1, we review the theory of new vector for \(GL_2\) and introduce the normalized Fourier coefficients \(\lambda_{\pi}(m)\) associated to a cuspidal representation \(\pi\). In Section 3.2, we recall the notion of Hecke operators, and in Section 3.3, we introduce the relevant Dirichlet series associated to a cuspidal representation of \(GL_2(\mathbb{A}_E)\). We prove an important identity of \(L\)-functions (cf. Lemma 3.3.1) that allows us to reduce our problem to the determination of the fibers of the Asai transfer. Finally, in Section 4, using the multiplicity one theorem for isobaric automorphic representations [11,10], we prove the main result alluded to at the beginning of this introduction. It is worth pointing out that it is sufficient to require that \(\lambda_{\pi}(\mathfrak{p}) = \lambda_{\pi'}(\mathfrak{p})\) for almost all prime ideals \(\mathfrak{p}\) of \(F\) (cf. Theorem 4.0.2).

A word about notation: Since \(E/F\) is a fixed quadratic extension, “Asai” or “tensor induction” is always taken with respect to \(E/F\). In Section 2, for any character \(\xi\), we write \(\xi_0\) for its restriction to \(C_F\) whenever it makes sense. Also, while writing the Hecke \(L\)-functions in this section, we sometimes include the field of definition as a subscript, for example \(L_k(s, \chi)\), where \(\chi\) is an idèle class character of \(k\).

2. The Asai transfer

We fix \(E/F\) a quadratic extension of number fields. For any number field \(k\), we write \(A_k\) for its ring of adèles and \(A_k^\times\) for its group of idèles. For a place \(u\) of \(k\), we write \(k_u\) to denote the completion of \(k\) at \(u\). Now, consider the algebraic group \(RE/GL_2\) given by the restriction of scalars; then its dual group \(^{t}(RE/GL_2)\) is given by \(GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rtimes \Gamma_F\), where \(\Gamma_F\) denotes the absolute Galois group of \(F\). There is a natural four-dimensional representation

\[ r: GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rtimes \Gamma_F \to GL(\mathbb{C}^2 \otimes \mathbb{C}^2) \]

given by

\[ r(\chi, \gamma, \gamma') = \begin{cases} (\chi \otimes \gamma) & \text{if } \gamma \text{ restricted to } E \text{ is trivial}, \\ (\gamma \otimes \chi) & \text{if } \gamma' \text{ restricted to } E \text{ is not trivial}. \end{cases} \]

This representation in turn yields an \(L\)-group homomorphism, also denoted as \(r\), from \(^{t}(RE/GL_2)\) to \(^{t}GL_4\). For each place \(v\) of \(F\), we let \(r_v\) denote the corresponding local \(L\)-group homomorphism obtained from \(r\) via restriction. Let \(W_{F_v}^r\) denote the Weil group if \(v\) is Archimedean, and the Weil–Deligne group if \(v\) is non-Archimedean.
Let $\pi$ be an irreducible cuspidal representation of $GL_2(\AA_E)$. Then we may consider $\pi$ as a representation of $R_{E/F}GL_2(\AA_F)$ and factorize it as a restricted tensor product, namely, $\pi = \otimes_v \pi_v$, where each $\pi_v$ is an irreducible admissible representation of $GL_2(E \otimes F_v)$. For every place $v$ of $F$, let $\phi_v : W'_{F_v} \to GL_2(\C) \times GL_2(\C) \times \Gamma_{F_v}$ be the local parameter attached to $\pi_v$ under the local Langlands correspondence. Now, for each place $v$ of $F$, let $AS(\pi_v)$ be the irreducible admissible representation of $GL_4(F_v)$ corresponding to the parameter $r_v \circ \phi_v$ under the local Langlands correspondence. Now, set $AS(\pi) := \otimes_v AS(\pi_v)$ – it is an irreducible admissible representation of $GL_4(\AA_F)$. By [12, Theorem 6.7], or see [17], one knows that $AS(\pi)$ is an isobaric automorphic representation of $GL_4(\AA_F)$. We refer to $AS(\pi)$ as the Asai transfer of $\pi$.

For each place $v$ of $F$, let $L(s, \pi_v, r_v)$ be the local $L$-function obtained by applying the Langlands–Shahidi method to the case $^2A_3 - 2$ (see [12, §4]), then by Propositions 6.2, 6.5, and Corollary 6.8 of [12], we have

$$L(s, \pi_v, r_v) = \frac{L(s, AS(\pi_v))}{\zeta_F(s)}$$

for all $v$. If $v$ splits as $(w_1, w_2)$ in $E$, then $L(s, AS(\pi_v))$ is the usual Rankin–Selberg $L$-function $L(s, \pi_{w_1} \times \pi_{w_2})$. Further, if $v$ is Archimedean, or if $v$ is unramified and $\pi_v$ is spherical, then it follows from the works of Shahidi [20,21] that the local factor $L(s, \pi_v, r_v)$ equals $L(s, r_v \circ \phi_v)$, where $L(s, r_v \circ \phi_v)$ is the local factor attached to the Weil representation $r_v \circ \phi_v$ [24]. (We also refer the reader to [12] for the relevant unramified calculations.)

If $v$ is a finite place that does not split in $E$, say $w|v$ is the unique place of $E$ lying over $v$, then $\pi_v (= \pi_w)$ is an irreducible unitary generic representation of $GL_2(E_w)$. It is well known that $\pi_v$ must be one of the following types: supercuspidal, Steinberg, or an irreducible principal series representation. If $\pi_v$ is supercuspidal or Steinberg, then $L(s, \pi_v, r_v)$ is computed by Goldberg [7, Theorem 5.2 and Theorem 5.6]. If $\pi_v$ is an irreducible principle series representation, namely, $\pi_v = L(\chi_1, \chi_2); \chi_1, \chi_2$, are characters of $E_w^*$, then using Shahidi’s result on multiplicativity of local factors [22] one checks that

$$L(s, \pi_v, r_v) = L_F(s, \chi_1) L_E(s, \chi_2) L_{E_w}(s, \chi_1 \chi_2),$$

where the local factors on the right-hand side are those of Hecke (cf. [24]). It is worth mentioning that for our purposes in this paper it suffices to just know the unramified computation of the local factors.

Next, we consider the case when $\pi$ is dihedral – our discussion in [12, §7] in this case is incomplete and warrants more explanation. So, from now on, unless mentioned otherwise, let us assume that $\pi$ is dihedral, i.e., $\pi = \pi \otimes \chi$, for some idèle class character $\chi$ of $E$. (For any number field $k$, we write $W_k$ to denote its global Weil group.) Then there is a two-dimensional representation $\sigma : W_E \to GL(V)$ such that

$$L(s, \pi) = L(s, \sigma),$$

where $L(s, \sigma)$ is the global $L$-function defined as in [24, §3]. In fact, $\sigma$ is irreducible and is induced from a character of an open subgroup $W_M \subset W_E$ of index 2, namely $\sigma = \text{ind}_{W_M}^{W_E} \mu$, $\mu^r \neq \mu$, $\tau \in \text{Gal}(M/E)$, $\tau \neq 1$ and $\pi$ is obtained by automorphic induction of $\mu$ [9]. (Note that $\mu$ can be identified with a character of $C_M$ via the reciprocity isomorphism $C_M \simeq W_M^{ab}$.) Using this $\sigma$, one can then define a four-dimensional representation $AS(\sigma)$ of $W_F$ in the following two ways:

First, we define $AS(\sigma)$ via tensor induction [4, §13]. (Recall for any number field $k$, the global Weil group $W_k$ comes equipped with a continuous homomorphism $\phi_k : W_k \to I_k$ with dense image.) Let us write $W_F = W_E \cup \wth W_E$, then $\phi_k(W_\theta)$ restricts to $\theta$ on $E$, where $\theta$ is the non-trivial element in $\text{Gal}(E/F)$. After fixing a basis for $V$, we can identify $GL(V)$ with $GL_2(\C)$ and hence we have the homomorphism $\sigma : W_E \to GL_2(\C)$. We then define the homomorphism
\[ \tilde{\sigma} : W_F \rightarrow \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \times \text{I}_F \]

over \( \text{I}_F \) by

\[ \tilde{\sigma}(x) = \begin{cases} 
(\sigma(x), \sigma(w_\theta xw_\theta^{-1}), \phi_F(x)) & \text{if } x \in W_E, \\
(\sigma(xw_\theta^{-1}), \sigma(w_\theta x), \phi_F(x)) & \text{if } x \notin W_E.
\end{cases} \]

We now let \( \text{As}(\sigma) = r \circ \tilde{\sigma} \).

Alternately, we may proceed as in [8, §4]. Namely, given an irreducible two-dimensional representation \( \sigma : W_E \rightarrow \text{GL}(V) \), consider \( V \) as a symplectic space via the usual determinant form

\[ \det : V \times V \rightarrow \mathbb{C}, \]

then identify \( V \) with \( V^\vee = \text{Hom}(V, \mathbb{C}) \) by \( v \mapsto l_v \); \( l_v(w) = \det(v, w) \); let \( \mathbb{V} = \text{End}(V) \), then we have the isomorphisms \( V \otimes V \cong V \otimes V^\vee \cong \mathbb{V} \) given by

\[ v \otimes w \mapsto v \otimes l_w \mapsto z \mapsto vl_w(z). \]

Let \( \phi \mapsto \iota \phi \) be the involution of \( \mathbb{V} \) corresponding to \( x \otimes y \mapsto -y \otimes x \); it is easy to verify that \( \phi \iota = \iota \phi \); now let \( q \) be the quadratic form on \( V \) defined by

\[ q(\phi, \psi) = \text{tr}(\phi \iota \psi). \]

Let \( \text{GO}(\mathbb{V}, q) = \{ g \in \text{GL}(\mathbb{V}) : q(g \phi, g \psi) = \lambda(g)q(\phi, \psi), \ \lambda(g) \in \mathbb{C}^\times \} \), then \( d(g) = \det(g) / \lambda(g)^2 \) defines a homomorphism from \( \text{GO}(\mathbb{V}, q) \rightarrow \{ \pm 1 \} \). We let \( \text{SGO}(\mathbb{V}, q) = \{ g \in \text{GO}(\mathbb{V}, q) : d(g) = 1 \} \).

Let us define the homomorphism \( \Phi : \text{GL}(V) \times \text{GL}(V) \rightarrow \text{SGO}(\mathbb{V}, q) \) by \( \Phi(\alpha, \beta) : \phi \mapsto \alpha \cdot \phi \cdot \iota \beta \); it is an exercise in elementary linear algebra to check that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{GL}(V) \times \text{GL}(V) & \xrightarrow{\Phi} & \text{SGO}(\mathbb{V}, q) \\
\text{det} \times \text{det} \downarrow & & \uparrow \lambda \\
\mathbb{C}^\times & \xrightarrow{\lambda} &
\end{array}
\]

and that the kernel of \( \Phi \) is \( \mathbb{C}^\times \), embedded in \( \text{GL}(V) \times \text{GL}(V) \) via \( x \mapsto (x, x^{-1}) \). Let \( \Theta \) denote the involution \( \phi \mapsto -\phi \). We can define \( \text{As}(\sigma) : W_F \rightarrow \text{GL}(\mathbb{V}) \) by

\[ \text{As}(\sigma)(x) = \begin{cases} 
\Phi(\sigma(x), \sigma(w_\theta xw_\theta^{-1})) & \text{if } x \in W_E, \\
\Phi(\sigma(xw_\theta^{-1}), \sigma(w_\theta x)) \Theta & \text{if } x \notin W_E.
\end{cases} \]

It is clear that the image of \( \text{As}(\sigma) \subset \text{GO}(\mathbb{V}, q) \) and that

\[ \text{As}(\sigma)\mid_{W_E} : W_E \rightarrow \text{SGO}(\mathbb{V}, q). \]

Further, if we take a basis \( \{ v_1, v_2 \} \) for \( V \) and then fix the basis

\[ \{ v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2 \} \]
for $V$, it can be verified that the matrix of $\Phi(\alpha, \beta)$ with respect to this basis is $M(\alpha) \otimes M(\beta)$, where $M(\alpha)$ (resp. $M(\beta)$) is the matrix of $\alpha$ (resp. $\beta$) with respect to $\{v_1, v_2\}$. Thus we see that the above two definitions of $\text{As}(\sigma)$ are in fact equivalent. In particular

$$\text{As}(\sigma)|_{W_E} \simeq \sigma \otimes \sigma^\theta,$$

where $\sigma^\theta$ is the representation of $W_E$ given by $\sigma^\theta(\kappa) = \sigma(w_\theta \kappa w_\theta^{-1})$. The argument in the following lemma is due to Dipendra Prasad.

**Lemma 2.0.1.** Suppose $\sigma, \sigma'$ are two-dimensional irreducible representations of $W_E$ such that $\text{As}(\sigma) \simeq \text{As}(\sigma')$, where the Asai representation is with respect to the quadratic extension $E/F$. Then there exists a character $\chi$ of $W_E$ such that $\sigma^\eta \simeq \sigma' \otimes \chi$, for some $\eta \in \text{Gal}(E/F) = \{1, \theta\}$.

**Proof.** Suppose $V$ is the representation space of $\sigma$; then from our discussions above, we have the exact sequence

$$1 \to \mathbb{C}^\times \to \text{GL}(V) \times \text{GL}(V) \to \text{SGO}(V, q) \to 1;$$

viewing this as an exact sequence of trivial $W_E$-modules we get the following exact sequence of pointed sets using group cohomology:

$$H^1(W_E, \mathbb{C}^\times) \to H^1(W_E, \text{GL}(V) \times \text{GL}(V)) \to H^1(W_E, \text{SGO}(V, q)).$$

Now, since $\sigma \otimes \sigma^\theta \simeq \sigma' \otimes \sigma'^{\theta}$, it follows from Lemma 5.2 of [1] that the elements determined by $(\sigma, \sigma^\theta)$ and $(\sigma', \sigma'^{\theta})$, respectively, have the same image in $H^1(W_E, \text{SGO}(V, q))$. Therefore, since $\mathbb{C}^\times$ is central in $\text{GL}(V) \times \text{GL}(V)$, they differ by an element of $H^1(W_E, \mathbb{C}^\times)$. In other words, the representations $(\sigma, \sigma^\theta)$ are unique up to twisting $\sigma \to \sigma \otimes \chi$, $\sigma^\theta \to \sigma^\theta \otimes \chi^{-1}$ with $\chi$ a character of $W_E$. \qed

If $\pi'$ is another cuspidal representation of $\text{GL}_2(\mathbb{A}_E)$ such that $\text{As}(\pi) \simeq \text{As}(\pi')$, by [12, Lemma 7.3], we know that $\pi'$ is also dihedral. Say $\sigma'$ is the corresponding two-dimensional representation of $W_E$ attached to $\pi'$. Then by hypothesis we have $\text{As}(\sigma) \simeq \text{As}(\sigma')$ and by restriction to $W_E$ we also have the equivalence $\sigma \otimes \sigma^\theta \simeq \sigma' \otimes \sigma'^{\theta}$. By Lemma 2.0.1, there exists an idèle class character $\chi$ of $E$ such that $\sigma' \otimes \chi \simeq \pi'$, $\gamma \in \text{Gal}(E/F)$. Then by moving to the automorphic side we get

$$\pi' \otimes \chi \simeq \pi'.$$

Now, with the details of the dihedral case in place, we recall [12, Theorem 7.1]:

**Theorem 2.0.2.** Suppose $\pi$ and $\pi'$ are cuspidal representations of $\text{GL}_2(\mathbb{A}_E)$ with $\text{As}(\pi) \simeq \text{As}(\pi')$. Then there is an idèle class character $\nu$ of $E$ such that

$$\pi' \otimes \nu \simeq \pi'$$

for some $\gamma \in \text{Gal}(E/F) = \{1, \theta\}$. If $\pi$ is non-dihedral, then $\nu|_{C_F} = 1$.

The main goal for the remainder of this section is to complete the description of the fibers in the dihedral case, namely, that $\nu$ can be chosen so that $\nu|_{C_F} = 1$, under the assumption that the central characters of $\pi$ and $\pi'$ agree on $C_F$. We adopt the following notation in what follows: For any quadratic extension $K/k$ of number fields and a cuspidal representation $\tau$ of $\text{GL}_n(\mathbb{A}_K)$, we write...
\(I^E_k(\tau)\) to denote the automorphic representation of \(GL_{2n}(\mathbb{A}_k)\) obtained via automorphic induction; for a cuspidal representation \(\tau\) of \(GL_n(\mathbb{A}_k)\), we write \((\tau)_k\) to denote the automorphic representation of \(GL_n(\mathbb{A}_k)\) obtained by base change transfer. (See [2].) We will use the basic properties of automorphic induction and base change (in the cyclic case), such as their fibers and cuspidality criterion, with no further mention. (See [16, Proposition 2.3.1] for a summary of these properties.) Also, we remind the reader that the notation \(\xi_0\) refers to the restriction of \(\xi\) to \(C_F\), whenever the notion makes sense.

First, we need the cuspidality criterion for \(As(\pi)\) when \(\pi\) is dihedral. As mentioned earlier the cuspidality criterion in [17,18] needs revision in the dihedral case and we refer the reader to Appendix A for a complete discussion of this topic. Here, for the sake of the reader's convenience we recall Theorem B, part (b) from Appendix A which is what we need for our purposes.

**Theorem 2.0.3.** (See Theorem B, part (b) from Appendix A.) Suppose \(\pi\) is a dihedral cuspidal representation of \(GL_2(\mathbb{A}_E)\). Then As(\(\pi\)) is not cuspidal if and only if \(\pi\) is induced from a quadratic extension \(M\) of \(E\) which is bi-quadratic over \(F\).

**Theorem 2.0.4.** Let \(E/F\) be a quadratic extension of number fields and let \(\pi, \pi'\) be dihedral cuspidal representations of \(GL_2(\mathbb{A}_E)\) whose central characters satisfy \(\omega_{\pi}|_{C_F} = \omega_{\pi'}|_{C_F}\). Suppose \(As(\pi) \simeq As(\pi')\). Then there exists an idèle class character \(v\) of \(E\) such that \(\pi' \otimes v \simeq \pi\) for some \(\gamma \in \text{Gal}(E/F)\), with \(v|_{C_F} = 1\).

**Proof.** From Lemma 2.0.1, we know that there exists an idèle class character \(\chi\) of \(E\) such that

\[
\pi \otimes \chi \simeq \pi', \quad \text{for some } \gamma \in \text{Gal}(E/F) = \{1, \theta\}.
\]

Since As(\(\pi'\)) \(\simeq As(\pi)\), we may suppose that \(\pi' \simeq \pi \otimes \chi\) by replacing \(\pi\) with \(\pi'\), if necessary, and our hypotheses on the central characters imply that \(\chi_0 = 1\). Let \(\Pi = As(\pi) \simeq As(\pi')\), it is an isobaric automorphic representation of \(GL_4(\mathbb{A}_F)\). A routine calculation with the Hecke–Frobenius parameters yields

\[
L^S(s, \Pi \times \tilde{\Pi}) = L^S(s, \pi \times \tilde{\pi}', R), \quad (2.0.2)
\]

where \(S\) is a finite set of places including the Archimedean ones so that all the relevant local data are unramified outside of \(S\). (This a natural identity since “tensor” or “Asai” induction commutes with taking tensor products and duals [4].) Here \(R\) is defined (similar to the definition of \(r\)) from \(GL_4(\mathbb{C}) \times GL_4(\mathbb{C}) \times I_F \rightarrow GL_1(\mathbb{C}) \times I_F\) via tensor induction. In what follows, we will also often use Lemma E from Appendix A (a basic result from class field theory) without explicitly mentioning it henceforth. We consider two main cases (A) and (B):

(A) \(\Pi\) is cuspidal: Then \(L(s, \Pi \times \tilde{\Pi})\) has a simple pole at \(s = 1\) [15]. On the other hand, we may compute \(L^S(s, \pi \times \tilde{\pi}', R)\) using \(\pi' \simeq \pi \otimes \chi\). We know that \(\pi \boxtimes \tilde{\pi}' = \text{Ad}(\pi)\chi^{-1} \boxplus \chi^{-1}\), where \(\text{Ad}(\pi) = \text{sym}^2(\pi) \otimes \omega_\pi^{-1}\) is an automorphic representation of \(GL_3(\mathbb{A}_E)\). In fact, since \(\pi\) is dihedral, say \(\pi = I^E_M(\mu), \text{Ad}(\pi)\) is given by

\[
\text{Ad}(\pi) = \tau \boxplus \delta, \quad \tau := I^E_M\left(\begin{array}{c} \mu \\ \mu \alpha \end{array}\right). \quad (2.0.3)
\]

where \(\alpha\) is the non-trivial Galois automorphism of \(M/E\), and \(\delta\) is the quadratic character attached to the extension \(M/E\) via class field theory. By Theorem 2.0.3, the cuspidality of \(\Pi\) implies that either \(M/F\) is not Galois in which case \(\delta\) is not \(\theta\)-invariant or \(M/F\) is a cyclic extension in which case \(\delta_0 = \delta_{E/F}\). In any event, we have

\[
\pi \boxtimes \tilde{\pi}' = \tau \boxplus \chi^{-1} \boxplus \chi^{-1} \delta \boxplus \chi^{-1}. \quad (2.0.4)
\]

Then \(L^S(s, \pi \times \tilde{\pi}', R)\) is given by
Here, the fourth $L$-function has a simple pole at $s = 1$ if and only if $\delta = \chi \chi^\theta$. However, $\delta \neq \chi \chi^\theta$ since $\chi_0^2 = 1$. Consequently this $L$-function is entire. We now consider the following two sub-cases.

(i) $\tau$ is cuspidal. If the first $L$-function in (2.0.5) has a simple pole at $s = 1$, then by a well-known result it follows that $\tau \otimes \chi$ is $\text{GL}_2(\mathbb{A}_F)$-distinguished. In particular, $\delta \chi^2$ the central character of $\tau \otimes \chi$ is trivial upon restriction to $\mathbb{A}_E^\times$, i.e., $\chi_0^2 \delta_0 = \delta_0 = 1$ which is not possible. Therefore $L^S(s, \text{As}(\tau) \otimes \chi_0^{-1})$ cannot have a simple pole at $s = 1$. Further, the second and the third $L$-functions in (2.0.5) are also entire. Thus, for (2.0.5) to have a simple pole at $s = 1$, precisely one of the characters $\chi_0, \chi_0^{-1} \delta_0$ must be trivial. If $\chi_0 = 1$, we are done. Otherwise $\chi_0^{-1} \delta_0 = 1$ in which case (since $\pi \simeq \pi \otimes \delta$) we may replace $\chi$ by $\chi \delta$ in $\pi' \simeq \pi \otimes \chi$. This completes the proof of the theorem with $\nu = \chi$ or $\nu = \chi \delta$.

(ii) $\tau$ is not cuspidal. Then $\frac{\mu}{\mu_F} = \eta \circ N_{M/E}$ for some idèle class character $\eta$ of $E$, and $\tau = \eta \boxplus \eta \delta$. Thus we get

$$\pi \boxtimes \pi' = \eta \chi^{-1} \boxplus \eta \delta \chi^{-1} \boxplus \delta \chi^{-1} \boxplus \chi^{-1},$$

and the right-hand side of (2.0.5) factors further into a product of

$$
\begin{align*}
L^S_F(s, \eta \delta \chi^{-1} \chi^{\theta}), \\
L^S_F(s, \eta \chi^{-1} \chi^{\theta}), \\
L^S_F(s, \eta \delta \chi^{-1} \chi^{\theta}), \\
L^S_F(s, \eta \delta \chi^{-1} \chi^{\theta}), \\
\delta \chi^{-1} \chi^{\theta}, \\
\delta \chi^{-1} \chi^{\theta}, \\
\eta \chi_0 \chi_0^{-1}, \\
\eta \delta \chi_0 \chi_0^{-1}, \\
L^S_F(s, \delta \chi_0 \chi_0^{-1}), \\
L^S_F(s, \chi_0^{-1}).
\end{align*}
$$

(2.0.7)

Once again, since either $\delta \neq \delta^{\theta}$ or $\delta_0 = \delta_{E/F} \neq 1$ and $\chi_0^2 = 1$, we see that $\delta \chi^{-1} \chi^{\theta}$ and $\eta \delta \chi^{-1} \chi^{\theta}$ are both non-trivial. Therefore $L^S(s, \eta \chi^{-1} \chi^{\theta})$ and $L^S(s, \eta \delta \chi^{-1} \chi^{\theta})$ are both entire. Further, since $\pi \simeq \pi \otimes \eta$ and $\pi$ is cuspidal, $\eta \neq 1, \delta$. In fact, $\pi$ is precisely induced from the three quadratic extensions of $E$ cut out by $\delta$, $\eta$, and $\eta \delta$.

We claim that the first four $L$-functions in (2.0.7) are entire. Otherwise, it is easy to obtain a contradiction. For example, say the first $L$-function has a simple pole at $s = 1$, then $\eta \delta^{\theta} = \chi \chi^\theta$ and consequently $\eta \delta_0 = \chi_0^2 = 1$, in particular the quadratic extension of $E$ cut out by $\eta \delta$ is bi-quadratic over $F$. Since $\pi$ is also induced from the quadratic extension cut out by $\eta \delta$ this contradicts the cuspidality of $\Pi$. Therefore we see that the right-hand side of (2.0.2) has a simple pole at $s = 1$ precisely when one of the last four $L$-functions in (2.0.7) has a simple pole. This in turn implies that exactly one of the characters $(\chi_0, \chi_0 \delta_0, \chi_0 \eta \delta_0, \chi_0 \eta \delta_0 \delta_0)$ is trivial. Since $\pi \simeq \pi \otimes \delta \simeq \pi \otimes \eta \simeq \pi \otimes \eta \delta$, we are done with the proof of the theorem in case (A) by taking $\nu \in \{\chi, \chi \delta, \chi \eta, \chi \eta \delta\}$.

(B) $\Pi$ is not cuspidal: By Theorem 2.0.3 we may assume $\pi$ is induced from a quadratic extension $M/E$ with $M/F$ bi-quadratic. In particular, $\delta^{\theta} = \delta$ and $\delta_0 = 1$. Let $\text{Gal}(M/E) = \{1, \alpha\}$ and let $\bar{\theta}$ denote the extension of $\theta$ to a Galois automorphism of $M/F$. Then $\text{Gal}(M/F) = \{1, \alpha, \bar{\theta}, \alpha \bar{\theta}\}$ with $\bar{\theta}^2 = 1$ and $\bar{\theta} \alpha = \alpha \bar{\theta}$. Moreover,
Moreover, with two sub-cases: 

\[ \tau_1 = I^E_M(\mu \mu^{\delta}), \quad \tau_2 = I^E_M(\mu \mu^{\delta \alpha}). \]

It is then straightforward to verify that the order of the pole 

\[ \text{ord}_{s=1} L^S(s, \Pi \times \tilde{\Pi}) = \begin{cases} 2, 3, & \text{if either } \tau_1 \text{ or } \tau_2 \text{ is cuspidal,} \\ 2, 3, 4, & \text{otherwise.} \end{cases} \]  

(2.0.9)

Here we remark that the order of the pole is dictated by the different possible isobaric types of \( \Pi \). Moreover, with \( \tau \) as in (2.0.3), we also have

\[ \text{As}(\tau)_E \simeq \tau \boxtimes \tau^0 = \tau_1' \boxplus \tau_2', \]  

(2.0.10)

where \( \tau_1' = I^E_M(\mu \mu^{\delta}), \quad \tau_2' = I^E_M(\mu \mu^{\delta \alpha}). \) Now, just as we did in case (A), we consider the following two sub-cases:

(i) \( \tau \) is cuspidal, i.e., \( (\mu^{\delta \alpha}_{\nu})^2 \neq 1 \). In particular, this implies that both \( \tau_1 \) and \( \tau_2 \) cannot fail to be cuspidal and hence (2.0.5) must have a pole of order 2 or 3 at \( s = 1 \). In fact it is clear that \( \Pi \) is either a sum of two cuspidal representations of \( GL_2(\mathbb{A}_F) \) or is an isobaric sum of type \((2, 1, 1)\); the order of the pole is 2 in the former case and it is 3 in the latter case. Observe that \( L^S(s, \text{As}(\tau) \otimes \chi_0^{-1}) \) cannot have a pole of order 2 or more, for if it did, \( \chi \chi^0 \) should appear at least twice in (2.0.10) which is not possible since \( \tau \) is cuspidal. If the order of the pole is 3, since \( \tau \) is cuspidal, it follows from (2.0.5) that \( \chi_0 = 1 \) and that \( \text{As}(\tau) \) contains the trivial representation. In fact, since

\[ L^S(s, \Pi \times \tilde{\Pi}) = L^S(s, \pi \times \tilde{\pi} \otimes R) = L^S_F(s, \text{As}(\tau))L^S_E(s, \tau^0)2L^S_E(s, \delta)\zeta(s)^2, \]

we see that the order of the pole at \( s = 1 \) is 3 if and only if \( \text{As}(\tau) \) contains the trivial representation.

Now, let us suppose (2.0.5) has a pole of order 2 at \( s = 1 \). Then, if \( \chi_0 \neq 1 \), \( L^S(s, \text{As}(\tau) \times \chi_0^{-1}) \) should have a simple pole at \( s = 1 \) with \( \delta = \chi \chi^0 \). This implies that \( \delta = \chi \chi^0 \) appears in either \( \tau_1' \) or \( \tau_2' \) (and not both) which in turn implies that either \( \mu \mu^{\delta} \) or \( \mu \mu^{\delta \alpha} \) is \( \alpha \)-invariant. Say, for example, \( \mu \mu^{\delta} = v \circ N_{M/E} \), then \( \pi^0 \simeq \pi \otimes \omega \) with \( \omega = v(\mu|_{\mathcal{E}})^{-1}\delta^{-1} \). Since

\[ \pi \boxtimes \pi^0 = \tau_2 \boxplus v \otimes v\delta \]

and (2.0.5) has a pole of order 2, we must have \( \nu^0 = v\delta \). From this it follows that \( \omega \omega^0 = \delta \).

Since \( (\pi')^0 \simeq \pi^0 \otimes \chi^{0} \simeq \pi \otimes \omega \chi^0 \), we may replace \( \pi' \) by \( \pi^0 \) and \( \chi \) by \( \chi^0 \omega \) in (2.0.5) and the equation remains valid. In particular, by the above argument this implies that \( \chi_{0\omega}^0 \) should be trivial.

(ii) \( \tau \) is not cuspidal, i.e., \( (\mu^{\delta \alpha}_{\nu})^2 = 1 \). Let \( \eta \) be as in (2.0.6), then as observed before, \( \pi \) is induced from three distinct quadratic extensions of \( E \) corresponding to the characters \( \delta, \eta \) and \( \eta \delta \). Then \( \tau_1 \) is cuspidal if and only if \( \tau_2 \) is cuspidal and this happens precisely when \( \mu^{\delta \alpha}_{\nu} \) is not \( \delta \)-invariant, in particular, \( \eta^{\delta} \neq \eta, \eta \delta \) and therefore \( \eta \eta^{\delta} \neq \delta \). Let us, for the moment suppose that both \( \tau_1 \) and \( \tau_2 \) are cuspidal, then \( \Pi \) is an isobaric sum of two cuspidal representations of \( GL_2(\mathbb{A}_F) \) and consequently \( L^S(s, \Pi \times \tilde{\Pi}) \) has a pole of order 2 at \( s = 1 \). Moreover (2.0.7) now holds with \( \delta^0 = \delta \) and \( \delta_0 = 1 \), in particular, the right-hand side of (2.0.7) is given by

\[ L^S_F(s, \chi_0)^2L^S_E(s, \eta_0 \chi_0)^2L^S_E(s, \delta \chi_0)\zeta^2(s, \delta \chi_0)\zeta^2(s, \delta \chi_0)^2 = (2.0.11) \]
Since \( \eta^0 \neq \eta \), the above expression has a pole of order 2 if and only if either \( \chi_0 = 1 \) or \( \chi_0 \eta_0 = 1 \). In either case we are done with the proof of our theorem.

So for the rest of the proof, let us suppose that \( \tau_1 \) (and hence \( \tau_2 \)) is not cuspidal. Then either \( \eta^0 = \eta \) or \( \eta^0 = \eta \delta \). If \( \eta^0 = \eta \) – namely, the three extensions corresponding to \( \pi \) are all Galois over \( F \) – then

\[
\text{ord}_{s=1} L^S(s, \Pi \times \widetilde{\Pi}) = \begin{cases} 2 & \text{if and only if } \eta_0 \neq 1 \text{ (and hence } = \delta_{E/F}), \\ 4 & \text{otherwise.} \end{cases} \tag{2.0.12}
\]

If the order of the pole is 4, i.e., \( \eta_0 = 1 \), it follows from (2.0.11) that \( \chi_0 = 1 \). On the other hand, if the order of the pole is 2, i.e., \( \eta_0 = \delta_{E/F} \), then once again from (2.0.11) it follows that exactly one of the characters \( \chi_0, \chi_0 \eta_0, \chi \chi^0 \delta \), is trivial. (Here, note that \( \chi \chi^0 \neq \eta, \eta \delta \) because \( \chi_0^2 = 1 \).) If \( \delta = \chi \chi^0 \), since \( \mu \mu^\delta \) is \( \alpha \)-invariant in the case at hand, we may argue just as we did in case (i) above. Finally, if \( \eta^0 = \eta \delta \), or in other words, the extensions corresponding to \( \eta \) and \( \eta \delta \) are not Galois over \( F \), then

\[
\text{ord}_{s=1} L^S(s, \Pi \times \widetilde{\Pi}) = 3.
\]

Then it follows from (2.0.11) that exactly one of the characters \( \chi_0, \eta_0 \chi_0 \), is trivial which is done with the proof of the theorem. \( \square \)

3. Automorphic representations of \( GL_2 \) and Dirichlet series

3.1. Fourier coefficients of cuspidal automorphic representations

Let \( \sigma \) denote the ring of integers of \( F \). For \( v < \infty \) we let \( \sigma_v \) denote the ring of integers of \( F_v \), \( p_v \) the unique prime ideal of \( \sigma_v \), \( \sigma_v \) a choice of generator of \( p_v \) which is normalized so that \( |\sigma_v|_v = q_v^{-1} \), where \( q_v \) is the cardinality of \( \sigma_v/p_v \). We will use either \( \sigma_v^{\times} \) or \( u_v \) for the group of local units.

The symbol \( \mathbb{A}_F, \mathbb{A}_{F,f} \) will denote the ring of finite adèles and \( F_\infty \) will denote \( \prod_{v|\infty} \mathbb{A}_F \) so that \( \mathbb{A}_F = \mathbb{A}_F \times \mathbb{A}_{F,f} \). (We let \( x_\infty \) and \( x_f \) denote the corresponding components of \( x \in \mathbb{A}_F \).) For any character \( \chi \) of \( \mathbb{A}_F \), we let \( \chi_f \) denote its restriction to \( \mathbb{A}_{F,f} \). We use the notation \( Cl_F \) to denote the class group of \( F \), in other words, it denotes the group of fractional ideals of \( F \) modulo the principal ideals. For any ideal \( a \) of \( F \), we write \( a^\times \) to denote the set of non-zero elements in \( a \). For any integral ideal \( m \) of \( F \), we write \( N(m) \), the norm of the ideal \( m \), to denote the index of \( m \) in \( \sigma \).

Let \( Z(\mathbb{A}_F) \) denote the center of \( GL_2(\mathbb{A}_F) \). For each place \( v \), we let \( Z(F_v) \) denote the center of \( GL_2(F_v) \). We fix \( K = \prod_v K_v \) of \( GL_2(\mathbb{A}_F) \), a maximal compact subgroup of \( GL_2(\mathbb{A}_F) \), where \( K_v = GL_2(\sigma_v) \). For \( v < \infty \), \( K_v = O(2, \mathbb{R}) \), if \( v \) is real, and \( K_v = U(2, \mathbb{C}) \) if \( v \) is complex. Let \( N \) be the group of upper triangular unipotent matrices in \( GL_2 \). We write \( N(\mathbb{A}_F) \) (resp. \( N(F_v) \)) to denote the corresponding \( \mathbb{A}_F \)-points (resp. \( F_v \)-points).

We fix the usual additive character \( \psi = \bigotimes_v \psi_v \) of \( F/\mathbb{A}_F \) whose conductor is the inverse different \( \delta^{-1} \) of \( F \). Namely, \( \psi = \psi_0 \circ tr \), where \( tr \) is the trace map from \( \mathbb{A}_F \) to \( \mathbb{A}_Q \), and \( \psi_0 \) is the standard additive character of \( Q/\mathbb{A}_Q \) which is unramified when restricted to \( Q_p, p \) a prime, and is \( e(x) \) when restricted to \( \mathbb{R} \). Let \( d \in \mathbb{A}_{F,f}^* \) be such that \( \psi_0^{ord_\mathbb{A}_F(d)} = 1 \). Let us now fix our choice of Haar measure on the idèle class group: For each finite place \( v \) of \( F \), let \( d^v x_v \) be the Haar measure on \( F_v^\times \) such that the volume of \( \sigma_v^{\times} \) is one. For \( v|\infty \), if \( d x_v \) is the ordinary Lebesgue measure if \( v \) is real and twice the ordinary Lebesgue measure if \( v \) is complex, we let \( d^x x_v = \frac{dx_v}{|\psi_v|_v} \). Then, our choice of local Haar measures yields a unique Haar measure \( d^x \) on \( \mathbb{A}_F^\times \) such that the volume of \( \prod_{v<\infty} \sigma_v^{\times} \) is one.

Let \( \pi \) be a cuspidal automorphic representation of \( GL_2(\mathbb{A}_F) \) with central character \( \omega_{\mathbb{A}_F} \). We write \( V_\pi \) to denote the space of \( \pi \). Note that \( V_\pi \) is a function space on \( GL_2(F) \backslash GL_2(\mathbb{A}_F) \). For each \( v \), let \( W(\pi_v, \psi_v) \) denote the \( \psi_v \)-Whittaker model of \( \pi_v \). We let \( W(\pi, \psi) = \bigotimes_v W(\pi_v, \psi_v) \) denote the corresponding global Whittaker model of \( \pi \).
We let \( c \in \mathbb{A}_F \) be the conductor of \( \pi \). We write \( c = \prod_{v < \infty} p_v^{m_v} \), where \( m_v = 0 \) whenever \( \pi_v \) is unramified and \( m_v > 0 \) otherwise. We let

\[
K_{1,v}(p_v^{m_v}) = \left\{ g \in \text{GL}_2(o_v) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod p_v^{m_v} \right\},
\]

and

\[
K_{0,v}(p_v^{m_v}) = \left\{ g \in \text{GL}_2(o_v) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod p_v^{m_v} \right\}.
\]

We then set \( K_1(c) = \prod_{v < \infty} K_{1,v}(p_v^{m_v}) \) and \( K_0(c) = \prod_{v < \infty} K_{0,v}(p_v^{m_v}) \). The definition of \( c \) is such that for each finite \( v \), the dimension of the space of \( K_{1,v}(p_v^{m_v}) \)-fixed vectors in \( \pi_v \) is one. Moreover, \( K_{0,v}(p_v^{m_v}) \) acts on this one-dimensional space by the central character \( \omega_{\pi_v} \), where \( \omega_{\pi_v} \) determines a character of \( K_{0,v}(p_v^{m_v}) \) by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \omega_{\pi_v}(d) \) if \( m_v > 0 \) and \( \omega_{\pi_v}(g_v) = 1 \) if \( m_v = 0 \). Observe that \( c \) determines an integral ideal of \( F \) which is also denoted as \( c \).

For every \( t \in \mathbb{A}_F^\times \), we denote by \( (t) \) the fractional ideal associated with \( t \). Let \( h \) be the number of ideal classes in the ideal class group \( \mathcal{C}_L \) of \( F \). We choose \( h \) elements \( t_1, \ldots, t_h \) in \( \mathbb{A}_F^\times \) so that \( (t_i)_\infty = 1 \) and so that the fractional ideals \( (t_i) \) form a complete set of representatives of such ideal classes. Set \( g_i = \begin{pmatrix} t_i & 1 \\ 1 & 1 \end{pmatrix} \), then by strong approximation theorem for \( \text{GL}_2 \) we have

\[
\text{GL}_2(\mathbb{A}_F) = \prod_{i=1}^h \text{GL}_2(F)g_i \mathbb{G}_\infty K_0(c).
\]

Here \( \mathbb{G}_\infty = \prod_{v < \infty} \text{GL}_2(F_v) \). There is a unique intertwining map from \( V_\pi \) onto \( W(\pi, \psi) \) which we will denote as \( \xi \mapsto W_\xi \). Every \( \xi \in V_\pi \) has a Fourier expansion

\[
\xi(g) = \sum_{y \in F^\times} W_\xi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} g \right) ; \quad \xi \in V_\pi, g \in \text{GL}_2(\mathbb{A}_F).
\]

For each finite \( v \), there is a unique \( \phi_v \in V_{\pi_v} \) for which the corresponding \( W_{\phi_v} \) is such that \( W_{\phi_v}(\begin{pmatrix} d_v & 1 \\ 0 & 1 \end{pmatrix}) = 1 \), and \( W_{\phi_v} \) transforms via the central character \( \omega_{\pi_v} \) for the action of \( K_0,v(p_v^{m_v}) \).

For \( v|\infty \), we pick \( \phi_v \in V_{\pi_v} \) such that

\[
\int_{F_v^\times} W_{\phi_v} \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) |y|_v^{s-\frac{1}{2}} d^\times y = L(s, \pi_v),
\]

where \( L(s, \pi_v) \) is the local \( L \)-function attached to \( \pi_v \). (See [9,6].) The function \( \phi = \bigotimes_v \phi_v \) is called the new vector of \( \pi \). If, for each place \( v \) of \( F \), \( L(s, \pi_v) \) denotes the local \( L \)-function attached to \( \pi_v \), let us form the completed \( L \)-function \( \Lambda(s, \pi) \) by setting \( \Lambda(s, \pi) = \prod_v L(s, \pi_v) \), then our choice of \( \phi \in V_\pi \) is such that

\[
N(\mathfrak{d})^{\frac{1}{2}-s} \int_{F^\times \setminus \mathbb{A}_F^\times} \phi \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) |y|^{1-s} \frac{1}{2} d^\times y = \Lambda(s, \pi).
\]

Let \( L(s, \pi) \) denote the finite part of \( \Lambda(s, \pi) \), i.e., the product over the finite places, we will express \( L(s, \pi) \) as a Dirichlet series. (It is worth pointing out that the notation \( L(s, \pi) \) is often used to denote
the completed $L$-function in the literature.) First note that the function $y \mapsto \phi_\xi \left( \left( \begin{array}{c} y \\ 1 \end{array} \right) \right)$ is invariant under $\prod_v a_v^\infty$. Further, since for each $v < \infty$, the choice of the measure $d^x y_v$ is such that $a_v^\infty$ has unit volume, we see that

$$A(s, \pi) = N(0)^{\frac{1}{2}-s} \int_{C(0)} \phi \left( \left( \begin{array}{c} y \\ 1 \end{array} \right) \right) |y|^{s-\frac{1}{2}} d^x y,$$

(3.1.1)

where $C(0) = F^\times \setminus A^\times_\infty / \prod_v a_v^\infty$.

Let $a(y)$ denote the matrix $\left( \begin{array}{c} y_1 \\ 1 \end{array} \right)$. Let us write

$$\phi \left( \left( \begin{array}{c} y \\ 1 \end{array} \right) \right) = \sum_{\gamma \in F^\times} a_\phi(y, \gamma) W_{\phi, \infty}(a(y_\infty y_\infty)),$$

where $a_\phi(y, \gamma) = \prod_{v < \infty} W_{\phi_v}(a(y_v y_v))$ and $W_{\phi, \infty} = \prod_{v|\infty} W_{\phi_v}$. Observe that $a_\phi(y, \gamma)$ only depends on the finite part $y_f$ of $y$. Moreover, for $v < \infty$, it is easy to see that $W_{\phi_v}(\left( \begin{array}{c} a \\ 1 \end{array} \right)) \neq 0$ for $a \notin \delta_v^{-1}$, where $\delta_v^{-1} = (d_v^{-1}) o_v$ is the conductor of $\psi_v$. Hence we have

$$\phi(a(y)) = \sum_{\gamma \in (y)^{-1} \delta^{-1} \times} a_\phi(y, \gamma) W_{\phi, \infty}(a(y_\infty y_\infty)),$$

(3.1.2)

where $(y)$ is the fractional ideal $(y) = \prod_{v < \infty} p_v^{\text{ord}_v(y_v)}$. Since $W_{\phi_v}(\cdot)$ is right invariant under $K_{1,v}(p_v^{m_v})$, it follows that

$$a_\phi(y, \epsilon \gamma) = a_\phi(y, \gamma), \quad \forall \epsilon \in o^\times.$$

Consider the equivalence relation $\sim$ in $(y)^{-1} \delta^{-1}$ which is given as follows: $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1 = \epsilon \gamma_2$, for some $\epsilon \in o^\times$. Then the right-hand side of (3.1.2) is given by

$$\sum_{\gamma \in \sim \setminus ((y)^{-1} \delta^{-1})^\times} a_\phi(y, \gamma) \sum_{\eta \in o^\times} W_{\phi, \infty}(a(\eta_\infty \gamma_\infty y_\infty)).$$

(3.1.3)

We have the exact sequence

$$1 \to o^\times \setminus F^\times_\infty \to C(o) \to Cl_F \to 1,$$

and by fixing the representatives $t_1, \ldots, t_h$, we have chosen a section of the last map. Using this we see that

$$\int_{C(0)} \phi \left( \left( \begin{array}{c} y \\ 1 \end{array} \right) \right) |y|^{s-\frac{1}{2}} d^x y = \int_{a^\times \setminus F^\times_\infty} \sum_j \phi \left( \left( \begin{array}{c} t_j y \\ 1 \end{array} \right) \right) |t_j y|^{s-\frac{1}{2}} d^x y;$$

which in turn by (3.1.3) yields
\[
A(s, \pi) = N(\mathfrak{o})^{\frac{1}{2} - s} \sum_{j=1}^{h} \left( \sum_{\gamma \in \sim \setminus (\mathfrak{a}_j)^{-1} \mathfrak{o}^{-1}} a_\phi(t_j, \gamma)|t_j|^{s - \frac{1}{2}} \right)
\times \int_{\mathcal{O} \times \mathcal{F}} \sum_{\eta \in \mathcal{O} \times \mathcal{F}} W_{\phi, \infty}(a(\eta_{\infty}y_{\infty}, y_{\infty}))|y_{\infty}|^{s - \frac{1}{2}} d^\times y_{\infty}.
\]

Effecting the change \( y_{\infty} \mapsto \gamma_{\infty}^{-1}y_{\infty} \) and combining the integral with the sum over \( \eta \) we get

\[
A(s, \pi) = N(\mathfrak{o})^{\frac{1}{2} - s} \left( \sum_{j=1}^{h} \sum_{\gamma \in \sim \setminus (\mathfrak{a}_j)^{-1} \mathfrak{o}^{-1}} a_\phi(t_j, \gamma)|t_j|^{s - \frac{1}{2}} |\gamma_{\infty}|^{-s + \frac{1}{2}} \right)
\times \int_{\mathcal{F}} W_{\phi, \infty}(a(y_{\infty}))|y_{\infty}|^{s - \frac{1}{2}} d^\times y_{\infty}.
\]

We define the normalized Fourier coefficients (of \( \pi \)) as follows: Given an integral ideal \( m \), there is a unique \( k, 1 \leq k \leq h \), such that \( m = (\gamma)a_k d \) for some \( \gamma \) in \( F \), we then set

\[
\lambda_{\pi}(m) = \sqrt{N((\gamma)(a_k)(\mathfrak{o}))}a_\phi(t_k, \gamma).
\]

Therefore, considering our choice of \( \phi_v, v|\infty \), and noting that \( |\gamma_{\infty}| = N(\gamma) \), we see that (3.1.4) takes the form

\[
A(s, \pi) = \left( \sum_{m \subset \mathcal{O}} \frac{\lambda_{\pi}(m)}{N(m)^s} \right) L(s, \pi_{\infty}).
\]

Here \( L(s, \pi_{\infty}) = \prod_{v|\infty} L(s, \pi_v) \). Therefore, we have expressed the finite \( L \)-function \( L(s, \pi) \) as a Dirichlet series, i.e.,

\[
L(s, \pi) = \sum_{m \subset \mathcal{O}} \frac{\lambda_{\pi}(m)}{N(m)^s}.
\]

3.2. Hecke operators

Here, we recall the definition of the Hecke operators. (See [23, §2] and the references therein.) Although the discussion in [23] is for totally real fields, the facts about the Hecke operators, namely, (3.2.3) and (3.2.4) below, remains valid for any number field. Keeping the notation of the previous section, let \( W = G_\infty \times K_0(c) \). Let us put

\[
Y = Y(c) = GL_2(A_F) \cap \left( G_\infty \times \prod_{v < \infty} Y(p_v^{m_v}) \right)
\]

with

\[
Y(p_v^{m_v}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_v); c \in p_v^{m_v}; a, b, d \in \mathcal{O}_v \right\}.
\]

Our definition of \( Y(p_v^{m_v}) \) is different from the one in [23], in the sense that the twisting by the local different is missing in our definition, instead we have normalized the Fourier coefficients \( \lambda_{\pi}(m) \)
accordingly with a twist by the different ideal \( d \). For \( y \in Y \) write \( W y W = \bigcup_j y_j W \) as a disjoint union of right \( W \)-cosets, and assume that \( (y_j)_\infty = 1 \) for each \( j \). For \( f \in V_\pi \) such that \( f(gk) = \omega_\pi(k)f(g) \), \( k \in K_0(c) \), we define a function \( f|_{W y W} \), which is again in \( V_\pi \) having the same \( K_0(c) \)-transformation property as \( f \), by

\[
(f|_{W y W})(x) = \sum_j \omega_\pi(y_j)^{-1} f(xy_j). \tag{3.2.1}
\]

This is independent of the choice of \( y_j \)'s. Here, we have extended the definition of \( \omega_\pi \) from \( K_0(c) \) to the finite part of \( Y \). In fact, if we choose the Haar measure \( dk = \prod_{v < \infty} dk_v \) on \( K_0(c) \) such that the volume of \( K_0, v(p^m_v) \) is one with respect to \( dk_v \) for all finite \( v \), then one can check that

\[
(f|_{W y W})(x) = \int_{K_0(c)} \omega_\pi^{-1}(k) f(xk) \, dk.
\]

For each non-zero integral ideal \( m \) of \( F \), we consider the operator \( T_c(m) \) acting on functions in \( V_\pi \) with the aforementioned \( K_0(c) \)-transformation property through

\[
T_c(m)f = N(m)^{-\frac{1}{2}} \sum_{i(\det(y)) = m} f|_{W y W}, \tag{3.2.2}
\]

where the sum is taken over distinct double cosets \( W y W \) with \( y \in Y \) such that \( i(\det(y)) = m \). Further, if \( m \) is prime to \( c \), we let \( S_c(m) \) be the operator defined by

\[
S_c(m)f = f|_{W a W},
\]

where \( a \) is any element of \( K_f^\times \) such that \( i(a) = m \) (the choice of \( a \) is irrelevant since, for each \( v \) with \( p_v \) prime to \( c \), \( f \) is right \( K_v \)-invariant); if \( m \) is not prime to \( c \), we define \( S_c(m) \) to be zero. Then one has the well-known relation (cf. [23, §2])

\[
T_c(m)T_c(n) = \sum_{a \supset m+n} S_c(a)T_c(a^{-2}mn) \tag{3.2.3}
\]

and we have the formal Euler product

\[
\prod_a T_c(m)Nm^{-s} = \prod_p (1 - T_c(p)Np^{-s} + S_c(p)Np^{-2s})^{-1}. \tag{3.2.4}
\]

It is known that the function \( \phi \) is a common eigenfunction for all the Hecke operators \( T_c(m) \) and \( S_c(m) \). In fact \( T_c(m)\phi = \lambda_\pi(m)\phi \) and \( S_c(m)\phi = \omega_\pi(m)\phi \) for all integral ideals \( m \). (Here, we note that \( \omega_\pi \) can also be viewed as a character of the group of fractional ideals [14, Chapter VII, §6].) Moreover, by virtue of (3.2.3), one has the useful formula:

\[
\lambda_\pi(m)\lambda_\pi(n) = \sum_{a \supset m+n} \omega_\pi(a)\lambda_\pi(a^{-2}mn). \tag{3.2.5}
\]
3.3. Dirichlet series

Let \( E/F \) be a quadratic extension of number fields. Suppose \( \pi = \otimes_w \pi_w \) is a unitary cuspidal representation of \( GL_2(\mathbb{A}_E) \) with central character \( \omega_{\pi} \); then one can attach two kinds of \( L \)-series to \( \pi \). First one is the standard \( L \)-series \( L(s, \pi) \) defined in Section 3.1 through the choice of a normalized new vector \( \phi_\xi \):

\[
L(s, \pi) = \sum_m \lambda_\pi(m)N(m)^{-s}
\]

where \( m \) runs over all integral ideals of \( E \). This converges for sufficiently large \( \Re(s) \), and can be continued to an analytic function on the whole \( s \)-plane; further it follows from (3.2.4) that

\[
L(s, \pi) = \prod_\wp \left( 1 - \lambda_\pi(\wp)X^{N(\wp)^{-s}} + \omega_{\pi}(\wp)X^{N(\wp)^{-2s}} \right)^{-1},
\]

where the product is over prime ideals \( \wp \) of \( E \). A second one is the \( L \)-series studied by Asai [3]:

\[
L_{Asai}(s, \pi) = L(2s, \omega_{\pi}|_F) \sum_{n \subseteq o_F} \lambda_\pi(n_o_E)N(n)^{-s}. (3.3.1)
\]

The main point here is that the second Dirichlet series (3.3.1) admits an Euler product over primes \( p \) of \( F \). Namely, for every prime ideal \( \wp \) of \( E \), let us factor

\[
X^2 - \lambda_\pi(\wp)X + \omega_{\pi}(\wp) = (X - \alpha_{\wp})(X - \beta_{\wp}).
\]

Lemma 3.3.1. The \( L \)-series \( L_{Asai}(s, \pi) \) is given by the Euler product

\[
L_{Asai}(s, \pi) = \prod_p P_p(N(p)^{-s}),
\]

where

\[
P_p(X)^{-1} = \begin{cases} 
(1 - \alpha_{\wp}^2X)(1 - \beta_{\wp}^2X)(1 - \alpha_{\wp}\beta_{\wp}X) & \text{if } p\wp E = \wp^2, \\
(1 - \alpha_{\wp}X)(1 - \beta_{\wp}X)(1 - \alpha_{\wp}\beta_{\wp}X) & \text{if } p\wp E = \wp, \\
(1 - \alpha_{\wp}\alpha_{\wp}'X)(1 - \alpha_{\wp}\beta_{\wp}'X)(1 - \beta_{\wp}\alpha_{\wp}'X)(1 - \beta_{\wp}\beta_{\wp}'X) & \text{if } p\wp E = \wp \wp'.
\end{cases}
\]

Proof. From (3.2.5), it follows that the coefficients \( \lambda_\pi(m) \) are multiplicative; then it follows that \( L_{Asai}(s, \pi) = \prod_p P_p(N(p)^{-s}) \), where the local factor \( P_p(N(p)^{-s}) \) is the product of the Euler factor of \( L(2s, \omega_{\pi}|_F) \) at \( p \) and \( \sum_{r \geq 0} \lambda_\pi(p^r\wp E)N(p)^{-rs} \). Now for any prime ideal \( \wp \) of \( E \), we have

\[
\sum_{r \geq 0} \lambda_\pi(\wp^r)X^r = (1 - \lambda_\pi(\wp)X + \omega_{\pi}(\wp)X^2)^{-1}.
\]

Then if \( p\wp E = \wp \), it follows that

\[
P_p(N(p)^{-s})^{-1} = (1 - \omega_{\pi}(\wp)N(p)^{-2s})(1 - \lambda_\pi(\wp)N(p)^{-s} + \omega_{\pi}(\wp)N(p)^{-2s})
\]

\[
= (1 - \alpha_{\wp}\beta_{\wp}N(p)^{-2s})(1 - \alpha_{\wp}N(p)^{-s})(1 - \beta_{\wp}N(p)^{-s}).
\]
Suppose \( p \circ E = \wp^2 \); then for \( r \geq 1 \), by virtue of (3.2.5), we have the relation
\[
\lambda_\pi (p \circ E) \lambda_\pi (p^r \circ E) = \lambda_\pi (p^{r-1} \circ E) \omega_\pi (\wp) + \lambda_\pi (p^{r+1} \circ E) + \omega_\pi (\wp) \lambda_\pi (p^r \circ E)
\]
which in turn implies that
\[
\sum_{r \geq 0} \lambda_\pi (p^r \circ E) X^r = \frac{1 + \omega_\pi (\wp) X}{1 - \lambda_\pi (p \circ E) X + \omega_\pi (\wp) X^2}.
\]  
(3.3.2)

We also have the relation \( \lambda_\pi (\wp)^2 = \lambda_\pi (\wp^2) + \omega_\pi (\wp) \) which readily follows from (3.2.5); since \( \alpha_\wp + \beta_\wp = \lambda_\pi (\wp) \), \( \alpha_\wp \beta_\wp = \omega_\pi (\wp) \), we see that (3.3.2) may be rewritten as
\[
\sum_{r \geq 0} \lambda_\pi (p^r \circ E) X^r = \frac{1 + \alpha_\wp \beta_\wp X}{(1 - \alpha_\wp^2 X)(1 - \beta_\wp^2 X)}.
\]

Since the Euler factor of \( L(2s, \omega_\pi | F) \) at \( p \) is \((1 - \alpha_\wp^2 \beta_\wp^2 N(p)^{-2s})^{-1} \), it follows that \( P_p (N(p)^{-s}) \) is given by the inverse of
\[
(1 - \alpha_\wp^2 N(p)^{-s})(1 - \beta_\wp^2 N(p)^{-s})(1 - \alpha_\wp \beta_\wp N(p)^{-s}).
\]

Finally, let us suppose that \( p \circ E = \wp \wp' \). The calculation here is lengthy but straightforward. Namely, by (3.2.5), for \( r \geq 2 \), we see that
\[
\lambda_\pi (p \circ E) \lambda_\pi (p^r \circ E) = \lambda_\pi (p^{r-1} \circ E) + \omega_\pi (p \circ E) \lambda_\pi (p^{r-1} \circ E)
\]
\[
+ \omega_\pi (\wp) \lambda_\pi (\wp^{-2} (p^{r+1} \circ E)) + \omega_\pi (\wp') \lambda_\pi (\wp'^{-2} (p^{r+1} \circ E))
\]

and that \( \lambda_\pi (\wp^{\prime^*}) \lambda_\pi (\wp') = \lambda_\pi (\wp^{\prime^*+1}) + \omega_\pi (\wp') \lambda_\pi (\wp^{\prime^*-1}) \), \( \wp^{\prime^*} = \wp, \wp' \); then using these relations, it can be verified that
\[
\sum_{r \geq 0} \lambda_\pi (p^r \circ E) N(p)^{-rs} = \frac{1 - \alpha_\wp \beta_\wp \alpha_\wp' \beta_\wp' N(p)^{-2s}}{(1 - \alpha_\wp \alpha_\wp' N(p)^{-s})(1 - \alpha_\wp \beta_\wp' N(p)^{-s})(1 - \beta_\wp \alpha_\wp' N(p)^{-s})(1 - \beta_\wp \beta_\wp' N(p)^{-s})}.
\]

(3.3.3)

This completes the proof of our lemma. \( \square \)

4. The main results

In this section we state our main results which essentially follows from our results in Section 2 and Section 3, respectively. Recall that the normalized Fourier coefficients \( \lambda_\pi (m) \) are precisely the Hecke eigenvalues (cf. Sections 3.1, 3.2). However, the reader should bear in mind that it is not always the case that Fourier coefficients are related to Hecke eigenvalues and the emphasis should be on Fourier coefficients in the general situation. (See the Introduction – paragraph 4 – and the references therein.)

**Theorem 4.0.2.** Let \( E/F \) a quadratic extension of number fields. Suppose \( \pi \) and \( \pi' \) are two unitary cuspidal representations of \( GL_2 (A_F) \) whose central characters are such that \( \omega_\pi | \zeta_F = \omega_\pi' | \zeta_F \). Let us also suppose that the Fourier coefficients of \( \pi \) and \( \pi' \) satisfy the condition \( \lambda_\pi (p) = \lambda_\pi' (p) \) for almost all prime ideals \( p \) of \( F \). Then either \( \pi \simeq \pi' \otimes \nu \) or \( \pi^0 \simeq \pi' \overline{\otimes} \nu \) for some idèle class character \( \nu \) of \( E \) such that \( (\nu | \zeta_F) = 1 \).
Proof. For any finite place \( v \) of \( F \) corresponding to a prime ideal \( p \), it follows from Lemma 3.3.1 and the description of the local factors \( L(s, \As(\pi_v)) \) that \( L(s, \As(\pi_v)) = \mathcal{P}_p(N(p)^{-s}) \). From our description of \( \mathcal{P}_p(N(p)^{-s}) \) in the proof of Lemma 3.3.1 (in terms of the Fourier coefficients of \( \pi_p \)), it follows that \( \As(\pi_v) \cong \As(\pi'_v) \) for almost all finite places \( v \) of \( F \). Since \( \As(\pi) \) and \( \As(\pi') \) are isobaric automorphic representations \([12]\), the usual multiplicity one theorem of Jacquet and Shalika \([11, 10]\) imply that \( \As(\pi) \cong \As(\pi') \). Now our result follows from Theorem 2.0.2 and Theorem 2.0.4.

The question was initially raised to the author in the context of Hilbert modular forms. For the sake of reference, we will include a version of Theorem 4.0.2 for Hilbert modular forms. Let us suppose that \( E/F \) is a quadratic extension of totally real fields. Let \( d \) be the degree of \( E \) over \( \mathbb{Q} \). Given \( k = (k_1, \ldots, k_d) \) a \( d \)-tuple of positive integers, and a finite order Hecke character \( \chi \) of \( E \), let \( \mathcal{S}_k(\chi) \) be the collection of all cuspidal Hilbert modular newforms of weight \( k \), which are common eigenfunctions for all the Hecke operators, and having central character \( \chi \). (We refer the reader to \([23, \S 2]\) for the definition and basic properties of Hilbert modular forms.) It is well known that \( f \in \mathcal{S}_k(\chi) \) corresponds to a cuspidal automorphic representation \( \pi(f) \) of \( \text{GL}_2(\mathbb{A}_E) \). Under this correspondence, the central character of \( \pi(f) \) is \( \chi | \cdot |^{k_0-1} \), where \( k_0 = \max\{k_i - 1\} \). Then the following is a consequence of Theorem 4.0.2:

Theorem 4.0.3. Let \( E/F \) be a quadratic extension of totally real fields. Let \( \chi \) and \( \chi' \) be finite order Hecke characters of \( E \) such that \( \chi|_{C_F} = \chi'|_{C_F} \). Suppose \( f \in \mathcal{S}_k(\chi) \), \( f' \in \mathcal{S}_k(\chi') \), are normalized newforms, defined with respect to the field \( E \), whose Fourier coefficients satisfy the relation

\[ c(n \mathfrak{o}_E, f) = c(n \mathfrak{o}_E, f') \]

for all non-zero integral ideals \( n \subset \mathfrak{o}_F \). Then there is a finite order Hecke character \( \nu \) of \( C_F \) such that \( \nu|_{C_F} = 1 \) and

\[ c(\gamma(m), f) = \nu(m)c(m, f') \quad \forall m \subset \mathfrak{o}_E \quad \text{for some } \gamma \in \text{Gal}(E/F). \]

Acknowledgments

This paper has an interesting history. An earlier draft of this paper was submitted for publication in the Shahidi volume on 14 May 2008 and was accepted for publication on August 1, 2008. However, due to some purely technical reasons, it was not included in the volume which came out in print in June, 2011. In the meantime (April, 2011 to be precise), Dipendra Prasad pointed out to the author that Dinakar Ramakrishnan’s cuspidality criterion \([17, 16]\) is incomplete in the dihedral case. Since then, both Dipendra Prasad and Dinakar Ramakrishnan have given different proofs of the revised cuspidality criterion which are included here as Appendix A. I have fixed the (earlier) proof of Theorem 2.0.4 by incorporating the revised cuspidality criterion.

I thank Dinakar Ramakrishnan for bringing the problem to my attention and for a number of useful comments pertaining to the main result of this paper, in particular, for explaining the importance of Fourier coefficients rather than Hecke eigenvalues. I am indebted to Dipendra Prasad for his crucial comments regarding the cuspidality criterion of the Asai transfer. In particular, thanks to both of them for providing Appendix A. I also thank Phil Kutzko and Freydoon Shahidi for useful discussions while writing this paper. It gives me great pleasure to dedicate this article to Freydoon Shahidi. I have been fortunate to have studied mathematics under his tutelage and I am grateful for his support and guidance over the years. Finally, I am grateful to James Cogdell for his courtesy.
Appendix A. On the cuspidality criterion for the Asai transfer to $GL(4)$

By Dipendra Prasad and Dinakar Ramakrishnan

A.1. Introduction

Let $F$ be a number field and $K$ a quadratic algebra over $F$, i.e., either $F \times F$ or a quadratic field extension of $F$. Denote by $G$ the $F$-group defined by $GL(2)/K$. Then, given any cuspidal automorphic representation $\pi$ of $G(\mathbb{A}_F)$, one has (cf. [A-8,A-9]) a transfer to an isobaric automorphic representation $\Pi$ of $GL_4(\mathbb{A}_F)$ corresponding to the $L$-homomorphism $L^1 G \to L^1 GL(4)$. Usually, $\Pi$ is called the Rankin–Selberg product when $K = F \times F$, and the Asai transfer when $K$ is a quadratic extension. (See also [A-5].) In the former case, $\pi$ is a pair $(\pi_1, \pi_2)$ of cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$, and $\Pi$ is denoted $\pi_1 \boxtimes \pi_2$, while in the latter case, $\Pi$ is denoted $\pi_K^1 \Pi$. The main purpose of this note is the following. In the appendix of [A-10], a criterion is given for deciding when $\Pi$ is cuspidal, which is correct for non-dihedral forms $\pi$, but has to be modified for the dihedral ones. This error was encountered in the key Asai case by the first author in his work with Anandavardhanan [A-1]. Here we give two different proofs of the corrected cuspidality criterion. The first one is due to the second author, which slightly modifies and adapts his original arguments in [A-8,A-10], while the second one, due to the first author, is different and may generalize to other situations. There is hardly any difference in the Rankin–Selberg case, so we give a unified single proof in that case. We hope that it is appropriate to present the proofs here as an appendix because Krishnamurthy also needs the corrected criterion for use in his work presented in [A-6].

The criterion has a natural analogue when $F$ is a local field, where by a cuspidal representation we will mean (as in [A-7]) a discrete series representation. In order to treat the local and global cases simultaneously, let us write $C_F$ for $F^*$, resp. the idèle class group $\mathbb{A}_F^\times/F^*$, in the former, resp. latter, case. In each case class field theory furnishes a natural isomorphism of $C_F$ with the abelianization of the Weil group $W_F$, and we will, by abuse of notation, use the same letter to denote the corresponding characters of $C_F$ and $W_F$. Let $A_n(F)$ denote the set of irreducible isobaric automorphic, resp. admissible, representations of $GL_n(\mathbb{A}_F)$, resp. $GL_n(F)$, when $F$ is global, resp. local. Then one knows (cf. [A-7A-4]) that for every $\pi$ in $A_n(F)$, there is a unique partition $n_1 + \cdots + n_r$ and cuspidal representations $\pi_j$ of $GL(n_j)$, $1 \leq j \leq r$, such that there is, in the sense of Langlands (cf. [A-7A-4]), an isobaric sum decomposition $\pi = \bigoplus_{j=1}^r \pi_j$, which means, in particular, that the $L$-function (resp. $\varepsilon$-factor) of $\pi$ is the product of the corresponding ones of the $\pi_j$. We will say that $\pi$ is of (isobaric) type $(n_1, \ldots, n_r)$, and call each $\pi_j$ an isobaric summand of $\pi$. We will normalize the order so that $n_i \geq n_j$ if $i \geq j$.

For convenience, we write down the full cuspidality criterion in the Rankin–Selberg and Asai situations, though a correction is needed only in the dihedral case. Recall that a cuspidal representation $\pi$ of $GL_2(\mathbb{A}_F)$ is dihedral iff $\pi \simeq \pi \otimes \delta$ for a quadratic character $\delta$ of $C_F$. In this case, if $E$ denotes the quadratic extension of $F$ over which $\delta$ becomes trivial, there is a character $\chi$ of $C_E$ such that $\pi$ is $I_E^E(\chi)$, the representation (automorphically) induced by $\chi$, implying, in particular, that $L(s, \pi) = L(s, \chi)$. When we began writing this appendix, we had the criterion that in the Asai case, $As_{K/F}(\pi)$ is cuspidal iff we have either (i) $\pi, \pi^b$ are twist equivalent, or (ii) $\pi$ is induced from a biquadratic extension $M$ of $F$ containing $K$. In the course of writing down the proof we realized that when $\pi$ is dihedral, (i) actually implies (ii), and the statement below reflects that. We thank Krishnamurthy for a helpful conversation regarding Lemma D in Appendix A.2.

We take this opportunity to give, in addition to the cuspidality criterion, the precise occurrence of the various possible isobaric types $(n_1, \ldots, n_r)$ of $\Pi$.

Theorem A. Let $F$ be a number field or a local field, and $\pi, \pi'$ cuspidal representations in $A_2(F)$. Denote by $\Pi$ the Rankin–Selberg product $\pi \boxtimes \pi'$, which is in $A_4(F)$. Then we have the following:

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1 School of Mathematics, TIFR, Mumbai, India 400005.

2 Department of Mathematics, Caltech, Pasadena, CA 91125, USA. Supported by the NSF grant DMS-1001916.
(a) If \( \pi \) or \( \pi' \) is non-dihedral, then \( \Pi \) is non-cuspidal iff \( \pi, \pi' \) are twist-equivalent, i.e., \( \pi' \simeq \pi \otimes \nu \), for an idèle class character \( \nu \) of \( F \).

(b) If \( \pi, \pi' \) are both dihedral, then \( \Pi \) is non-cuspidal iff they are both induced from a common quadratic extension.

(c) \( \Pi \) is of type \((3, 1)\) iff \( \pi, \pi' \) are twist equivalent and non-dihedral, while it is of type \((2, 1, 1)\), resp. \((1, 1, 1, 1)\), iff \( \pi, \pi' \) are twist equivalent and dihedral, both induced from a unique, resp. non-unique, quadratic extension. \( \Pi \) is of type \((2, 2)\) iff \( \pi, \pi' \) are not twist equivalent, but are both dihedral, induced from a common quadratic extension.

Now we turn to the statement of the result in the Asai situation.

**Theorem B.** Let \( F \) be a number field or a local field, \( K/F \) a quadratic extension with non-trivial automorphism \( \theta \), and \( \pi \) a cuspidal representation in \( \mathcal{A}_2(K) \). Denote by \( \Pi \) the Asai transfer \( \text{As}_K/F(\pi) \), which is in \( \mathcal{A}_4(F) \). Then we have the following:

(a) If \( \pi \) is non-dihedral, then \( \Pi \) is non-cuspidal iff \( \pi \) and \( \pi^\theta \) are twist-equivalent.

(b) If \( \pi \) is dihedral, then \( \Pi \) is non-cuspidal iff \( \pi \) is induced from a quadratic extension \( M \) of \( K \) which is biquadratic over \( F \).

(c) \( \Pi \) admits an isobaric summand \( \chi \) in \( \mathcal{A}_4(F) \) iff \( \pi^\theta \simeq \pi^\nu \otimes \chi \), for a \( \theta \)-invariant character \( \chi \); in this situation, \( \Pi \) is of type \((3, 1)\) iff it is non-dihedral, while it is of type \((2, 1, 1)\), resp. \((1, 1, 1, 1)\), iff it is dihedral, induced from a unique, resp. non-unique, quartic Galois extension \( M \) of \( F \) containing \( K \).

(d) \( \Pi \) is of type \((2, 2)\) iff \( \pi \) is dihedral, induced from a biquadratic extension of \( F \), and there is no \( \theta \)-invariant character \( \chi \) occurring as an isobaric summand of \( \pi \otimes \pi^\theta \).

The first assertion of part (c) can also be deduced from [A-3].

**A.2. The Rankin–Selberg case: Proof of Theorem A**

When \( \pi, \pi' \) are both non-dihedral, the proof of (a) is given in [A-10] in the global case, which also works in the local non-Archimedean case when \( \pi, \pi' \) are both supercuspidal. Suppose \( \pi \) is special, with \( F \) local non-Archimedean, with parameter \( \sigma = \text{St}(v) : W_F \times \text{SL}(2, \mathbb{C}) \to \text{GL}(2, \mathbb{C}) \), \((w, g) \mapsto v(w)g\), where \( v \in \mathcal{A}_1(F) \). If \( \pi^\nu \) is supercuspidal, which need not be dihedral, with parameter \( \sigma^\nu : W_F \to \text{GL}_2(\mathbb{C}) \), then by [A-8], \( \Pi \) is the generalized special representation, hence square-integrable, of \( \text{GL}_4(F) \) with parameter \( \sigma \otimes \sigma^\nu \), which is irreducible. If \( \pi, \pi' \) are both special, say of the form \( \text{St}(v), \text{St}(v') \) for characters \( v, v' \) of \( F \), then \( \pi^\nu \simeq \pi \otimes v' v^{-1} \), and \( \Pi \) is non-cuspidal of type \((3, 1)\). Note that the case \( F = \mathbb{C} \) does not occur here, as there is no cuspidal element of \( \mathcal{A}_2(F) \), and if \( F = \mathbb{R} \), \( \pi, \pi' \) are both dihedral. So we may disregard the Archimedean case for part (a).

When \( \pi' \) is dihedral, say of the form \( \iota_F(\chi) \), we have

\[
\Pi \simeq \iota_F(\pi_K \otimes \chi),
\]

where \( \pi_K \) is the base change of \( \pi \) to \( \text{GL}(2)/K \), which is cuspidal since \( \pi \) is non-dihedral. By [A-2], the (automorphically) induced representation \( \Pi \) is non-cuspidal iff \( \pi_K \otimes \chi \) is invariant under the non-trivial automorphism \( \theta \) of \( K/F \), i.e., iff

\[
\pi_K \simeq \pi_K \otimes \lambda, \quad \text{with } \lambda = \chi^\theta / \chi,
\]  

(\text{A.2.1})

where \( \chi^\theta \) denotes \( \chi \circ \theta \), resp. \( \chi^{|\theta|} : w \mapsto \chi(\theta w \theta^{-1}) \), if viewed as a character of \( \mathcal{C}_K \), resp. \( W_K \); here \( \theta \) denotes any element of \( W_F \) which lifts \( \theta \in W_{K/F} \). Taking central characters, we see that \( \lambda^2 = 1 \) when (A.2.1) holds, and moreover, \( \lambda^\theta = \lambda^{-1} \). Hence \( \lambda \) is \( \theta \)-invariant and we may write \( \lambda \) as the base change \( v_K = v \circ N_{K/F} \), for a character \( v \) of \( C_F \). Then \( \pi \) and \( \pi \otimes v \) have the same base change to \( \text{GL}(2)/K \), and since \( \pi_K \) is cuspidal, \( \pi \) is isomorphic to either \( \pi \otimes v \) or to \( \pi \otimes (v \delta) \), where \( \delta \) is the quadratic character of \( C_F \) attached to \( K/F \). In either case, noting that \( \delta \neq v \) as \( v_K = \lambda \neq 1 \), we see that \( \pi \) admits
a non-trivial quadratic twist, contradicting its non-dihedral nature. So $\Pi$ must be cuspidal if exactly one of $\{\pi, \pi'\}$ is dihedral. This finishes the proof of part (a).

Let us now turn to (b), and assume that $\pi, \pi'$ are both dihedral. If they are induced from a common quadratic extension $K$, i.e., respectively of the form $I_K^E(\chi)$ and $I_K^F(\chi')$, it follows by Mackey that we have the isobaric sum decomposition

$$\Pi = I_K^E(\chi) \boxplus I_K^F(\chi') \simeq I_K^E(\chi \chi') \boxplus I_K^E(\chi' \chi').$$

(A.2.2)

Hence $\Pi$ is not cuspidal in this case.

We need to prove the converse, which leads us to the situation when $\pi = I_E^F(\nu)$ and $\pi' = I_K^E(\mu)$, for characters $\nu, \mu$ of distinct quadratic extensions $E, K$ of $F$. The assertion in this case is that if $\Pi := \pi \boxtimes \pi'$ is not cuspidal, then $\pi, \pi'$ are both induced from a common quadratic extension $M$ of $F$.

We have

$$\Pi \simeq I_K^E(\pi_K \otimes \mu),$$

(A.2.3)

where $\pi_K$ denotes the base change of $\pi$ to $K$. If $\pi_K$ is not cuspidal, then $\pi, \pi'$ are both induced from $K$, and we are done. So we may assume that $\pi_K$ is cuspidal. Similarly, we may assume that $\pi_K'$ is cuspidal.

Let $\theta$ be the non-trivial automorphism of $K/F$. When $\Pi$ is not cuspidal, $\pi_K \otimes \mu$ must be $\theta$-invariant, yielding

$$\pi_K \simeq \pi_K \otimes \lambda, \quad \text{with } \lambda = \mu^\theta / \mu.$$  

(A.2.4)

The character $\lambda$ must be quadratic as seen by comparing central characters. (If $\lambda$ were trivial, $\pi'$ would not be cuspidal.) Now, since $\theta^2 = 1$,

$$\lambda^\theta = \mu / \mu^\theta = \lambda^{-1} = \lambda.$$  

Hence $\lambda$ is $\theta$-invariant, hence is the pull-back by norm of a character $\xi$ of $F$. Thus

$$\pi_K \simeq (\pi \otimes \xi)_K.$$  

Then

$$\pi \simeq \pi \otimes \xi \quad \text{or} \quad \pi \simeq \pi \otimes \xi \delta,$$  

(A.2.5)

where $\delta$ the quadratic character of $F$ attached to $K/F$. Since $\delta \neq 1$, $\xi$ and $\xi \delta$ are distinct. Moreover, since $\xi_K = \lambda \neq 1, \xi \neq 1, \delta$.

Let $M, M'$ be the quadratic extensions of $F$ corresponding to $\xi, \xi \delta$ respectively. Then $\pi$ is induced from either $M$ or $M'$.

Next we work with $\pi'$. The central character $\omega'$ of $\pi'$ is $\delta$ times the restriction $\mu_0$ of $\mu$ to $F$. Also,

$$\text{sym}^2(\pi') = I_K^E(\mu^2) \boxplus \mu_0.$$  

(A.2.6)

Since $\lambda = \xi \circ N_{K/F}, \lambda^2 = 1 = (\mu / \mu^\theta)^2$, so $\mu^2$ is $\theta$-invariant. Thus

$$\text{Ad}(\pi') = \text{sym}^2(\pi') \otimes \omega'^{-1} = \xi \boxtimes \xi \delta \boxplus \delta.$$  

(A.2.7)

Let us, by abuse of notation, use $\pi'$ to also denote the associated 2-dimensional dihedral representation of the Weil group $W_F$. Then
which, when restricted to either of three different quadratic extensions, namely $K$, $M$ and $M'$, contains the trivial representation twice. This implies that the restriction of $\pi'$ to any of these three quadratic extensions is reducible; hence $\pi'$ is induced from every one of these extensions. On the other hand, since $M/K$ (resp. $M'/K$) is cut out by $\xi$ (resp. $\xi\delta$), (A.2.5) implies that $\pi$ is induced from either $M$ or $M'$. Thus $\pi$ and $\pi'$ are both induced by a common quadratic extension, completing the proof of part (b).

Now on to part (c). If $\Pi = \pi \boxtimes \pi'$ is non-cuspidal, then we have seen that $\pi, \pi'$ are both non-dihedral or both dihedral. The possible isobaric types are $(3, 1), (2, 2), (2, 1, 1)$, and $(1, 1, 1, 1)$. Clearly, a $GL(1)$ factor is a summand iff $\pi, \pi'$ are twist equivalent. Moreover, there is more than one character of $C_F$ appearing in the isobaric sum decomposition iff $\pi$, and hence $\pi'$, admits a self-twist by a non-trivial character, i.e., they are dihedral. When $\pi, \pi'$ are both non-dihedral, we know (by part (a)), that $\Pi$ is non-cuspidal iff $\pi' \simeq \pi \otimes \omega$, for some $\omega \in \mathcal{A}_3(F)$, which gives

$$\Pi \simeq (\text{sym}^2(\pi) \otimes \nu) \boxplus \omega\nu,$$

(A.2.8)

where $\omega$ is the central character of $\pi$ and $\text{sym}^2(\pi)$ is the symmetric square representation in $\mathcal{A}_3(F)$ attached to $\pi$, which is cuspidal by Gelbart–Jacquet since $\pi$ is non-dihedral. Hence (A.2.8) is an isobaric sum decomposition of type $(3, 1)$. Consequently, the isobaric type $(2, 2)$ occurs iff $\pi, \pi'$ are dihedral and not twist equivalent. □

A.3. The Asai case: The first proof of Theorem B

Let $(K/F, \theta, \pi)$ be as in Theorem B, with $\Pi$ denoting the Asai transfer $\mathcal{A}_{K/F}(\pi)$ in $\mathcal{A}_3(F)$. Then the base change $\Pi_K$ to $GL(4)/K$ is (isomorphic to) the Rankin–Selberg product $\pi \boxtimes (\pi \circ \theta)$. When $F$ is global, or when $F$ is local with $\pi$ non-dihedral and non-special, a proof of the assertion of part (a) of Theorem B is in [A-10]. If $F$ is non-Archimedean and $\pi$ is special attached to $\nu \in \mathcal{A}_1(K)$, then $\pi \circ \theta$ is also special, attached to $\nu\theta$, hence twist equivalent to $\pi$, and $\Pi_K = \pi \boxtimes (\pi \circ \theta)$ is non-cuspidal of type $(3, 1)$.

Let us now turn to the proof of part (b), so that $\pi$ is a dihedral representation $I^K_E(\psi)$, where $\psi$ is a character of $C_E$. Again we will, by abuse of notation, identify $\pi$ with the corresponding 2-dimensional induced representation of $W_K$, which is irreducible as $\pi$ is cuspidal. So we may also think of $\mathcal{A}_{K/F}(\pi)$ as a 4-dimensional representation of $W_F$. Note also that in this (dihedral) case, if $\tilde{\theta}$ is an element of $G_F = \text{Gal}(F/F)$ lifting $\theta$, $\pi \circ \theta$ is also dihedral of the form $I^K_{E^\theta}(\psi^{\theta})$, where $E^\theta = \tilde{\theta}(E)$, and $\psi^{\theta}(x) = \psi(\tilde{\theta}x\tilde{\theta}^{-1})$, for all $x$ in $E_{E^\theta}$.

Lemma C. Let $(\pi, K/F, \theta)$ be as above, with $\pi$ dihedral. Then $\Pi_K \simeq \pi \boxtimes (\pi \circ \theta)$ is non-cuspidal iff $\pi$ and $\pi \circ \theta$ are induced from a common quadratic extension $M/K$ which is Galois over $F$.

Proof. By Theorem A we know that the non-cuspidality of $\Pi_K$ is equivalent to both $\pi$ and $\pi \circ \theta$ being induced from a common quadratic extension $M/K$. If $M$ is the unique quadratic extension of $K$ inducing $\pi$, then it is necessarily Galois over $F$ as $\pi \circ \theta$ is induced from a Galois conjugate of $M$. If, on the other hand, $\pi$ is induced from more than one, hence three, quadratic extensions $E_j$, $j \in \{1, 2, 3\}$, of $K$ cut out by quadratic characters $\delta_j$ of $W_K$, necessarily with $E_1E_2 = E_1E_3 = E_2E_3$, then $\pi \circ \theta$ is induced from each $E_j^\theta$, $1 \leq j \leq 3$. We must have $M = E_i = E_j^\theta$, for some $i, j \leq 3$. If $i = j$, $M$ itself is Galois over $F$, and we are done. So let $i \neq j$. Then $E_j^\theta = E_i$, and the remaining third quadratic extension $E_k$, say, must be preserved by $\theta$ and hence Galois over $F$. It is also a common inducing field for $\pi$ and $\pi \circ \theta$. □

Now let $M$ be a quartic Galois extension of $F$ containing $K$, with $\pi = I^K_M(\lambda)$. Then $\pi^\theta = I^K_M(\lambda^{\tilde{\theta}})$, where $\tilde{\theta}$ is a lift of $\theta$ to $\text{Gal}(M/F)$. Denote by $\tau$ the non-trivial element of $\text{Gal}(M/K)$. Then, as we
have seen in Appendix A.1, one has by Mackey,

$$\Pi_K \simeq \pi \boxtimes (\pi \circ \theta) \simeq V \boxplus V', \quad \text{where } V = I^K_M(\lambda \lambda \delta), \quad V' = I^K_M(\lambda \tau \lambda ^{\hat{\delta}}).$$  \hspace{1cm} (A.3.1)

Consequently, the base change of $\Pi_M$ of $\Pi$ to $M$ is an isobaric sum of the members of the following set:

$$S := \{\lambda \lambda \delta, \lambda ^{\tau \lambda ^{\hat{\delta}}}, \lambda ^{\tau \lambda ^{\hat{\delta}}}, \lambda ^{\lambda \delta}\}. \hspace{1cm} (A.3.2)$$

When $M/K$ is bi-quadratic, $\delta$ has order 2, and preserves $V_M \simeq \lambda \lambda \delta \boxplus \lambda ^{\tau \lambda ^{\hat{\delta}}}$. It follows that $V^\theta \simeq V$, and $\Pi$ is non-cuspidal (and not of type $(3, 1)$). Part (b) of Theorem B will be proved if we establish the following

**Lemma D.** Suppose $M/F$ is cyclic. Then $\Pi$ is non-cuspidal iff $\pi^\theta$ is twist equivalent to $\pi$, and in this case, $\pi$ is also induced from a biquadratic extension $M'$ of $F$ containing $K$.

**Proof.** Note that the set $S$ above has 2 or 4 elements, and that $\Pi$ is cuspidal when $|S| = 4$ and $\text{Gal}(M/F)$ has a unique orbit in $S$. Since $M/F$ is cyclic, $\delta$ is of order 4, with $\theta = \delta^2$. $S$ has two elements iff $\lambda \lambda \delta$ is $\tau$-invariant, i.e., $\lambda \lambda \delta = \chi_M$, for some character $\chi$ of $C_K$. Hence $I^K_M(\lambda \delta) \simeq I^K_M(\lambda^{-1} \chi_M)$, implying that when $\Pi$ is non-cuspidal,

$$\pi^\theta \simeq \pi \chi \otimes \chi.$$  \hspace{1cm} (A.3.3)

Conversely, suppose (A.3.3) holds. Then $\Pi_K$ contains a character, hence $\Pi$ must contain a character or a two-dimensional subrepresentation of $W_F$, completing this part of Lemma D.

It remains to show that when (A.3.3) holds, $\pi$ is induced from a biquadratic extension as well. For any character $\xi$ of $C_K$, let $\xi_0$ denote its restriction to $C_F$. Put $\chi' = \omega^{-1} \chi$, so that $\pi^\theta \simeq \pi \otimes \chi'$. Taking central characters, we get $\omega^\delta = \omega \chi'^2$ implying that $\chi_0'^2 = 1$, and $(\chi'/\chi')^2 = 1$. If $\chi_0'$ is quadratic and unequal to $\delta$, the quadratic character of $C_F$ attached to $K$, then $\pi$ will be induced from the biquadratic extension $E_K$, where $E$ is the extension of $F$ cut out by $\chi_0'$. If $\chi_0' = 1$, then $\chi' = \omega^{-1} \chi$ is a quadratic character of $C_F$ corresponding to $M$, and $\epsilon = \nu K$ the quadratic character of $C_K$ corresponding to the quadratic extension $M$ of $K$. Since $\pi \simeq \pi \otimes \epsilon$, we have $\eta K \simeq \eta K \otimes \epsilon$, and so $\eta K \simeq \eta K \otimes \nu K$. Since $\eta$ is cuspidal, $\eta \otimes \nu$ is isomorphic to $\eta$ or $\eta \otimes \delta$. Either way, by taking central characters, get $\nu / \delta$, which gives a contradiction. So we may assume that $\chi_0' = \delta$. Then, $\text{As}(\pi) \simeq \text{As}(\pi^\theta) \simeq \text{As}(\pi) \otimes \chi_0'$, and so $\text{As}(\pi)$ admits a self-twist by $\delta$. Then, if any character $\nu$ of $C_F$ occurs in the isobaric sum decomposition of $\text{As}(\pi), \nu \delta$ will also occur in $\text{As}(\pi)$. Hence over $K$, $\pi \boxtimes \pi^\theta$ will contain $\nu K$ with multiplicity 2, which contradicts the cuspidality of $\pi$. So $\theta$ must move $\chi$, and $\xi := \chi'/\chi'$ is a quadratic character of $C_K$ fixed by $\theta$ such that $\xi_0 = 1$. Such a $\xi$ necessarily cuts out a biquadratic extension $M'$ of $F$ containing $K$, since otherwise $M'$ would be cyclic and $\xi = \beta K$ for a quadratic character $\beta$ of $C_F$, implying that $\xi_0 = \beta^2 \neq 1$. Since $\xi$ appears in $\pi \boxtimes \pi^\chi$, $\pi$ is induced from (the biquadratic extension) $M'$.

Now on to the proof of part (c). We may assume that $\Pi = \text{As}_{K/F}(\pi)$ is not cuspidal. Recall that $\pi^\chi \simeq \pi \otimes \omega^{-1}$, and $\text{Ad}(\pi) \simeq \text{sym}^2(\pi) \otimes \omega^{-1}$, where $\omega$ is the central character of $\pi$. If $\pi$ is not dihedral, then $\Pi_K = \pi \boxtimes (\pi \circ \theta)$ is non-cuspidal iff (A.3.3) holds for some $\chi \in A_1(K)$, in which case

$$\Pi_K \simeq (\text{Ad}(\pi) \otimes \chi) \boxplus \chi,$$  \hspace{1cm} (A.3.4)

which is isobaric of type $(3, 1)$ (since sym$^2(\pi)$ is cuspidal for non-dihedral $\pi$). Since $\text{Gal}(K/F)$ must preserve the two summands in this case, $\text{As}_{K/F}(\pi)$ is also of type $(3, 1)$, and $\chi$ is forced to be $\theta$-invariant.
It is left to focus on the case when \( \pi \) is dihedral with \( \Pi \) non-cuspidal. Then by (b), there is a biquadratic extension \( M \) of \( F \) containing \( K \) with non-trivial automorphism \( \tau \) of \( M/K \), such that \( \pi = I_M^K(\lambda) \) and (A.3.1) holds, with \( \hat{\theta}^2 = 1 \) and \( \hat{\theta} = \tau \hat{\theta} \). Note that \( \omega = \lambda_0 \epsilon_K \), where \( \lambda_0 \) is the restriction of \( \lambda \) to \( C_K \) and \( \epsilon \) is a quadratic character of \( C_F \) such that \( M \) is cut out over \( K \) by \( \epsilon_K = \epsilon \circ N_{K/F} \). From the proof of Theorem A we know that \( \Pi_K \) is of type \((2, 2)\) iff \( \pi \) is not twist equivalent to \( \pi^\theta \), and in this case \( \Pi \), being non-cuspidal, is also of type \((2, 2)\). So we may assume that (A.3.3) holds as well, which implies, since \( (\lambda_0)_M = \lambda \lambda^\tau \), that \( \lambda^\delta \in \{ (\lambda^\tau)^{-1} \chi_M, \lambda^{-1} \chi_M \} \). So (A.3.1) yields, since \( I_M^K(\chi_M) \simeq \chi \boxtimes \chi_{\epsilon_K} \).

\[
\Pi_K \simeq W \boxplus W', \quad \text{where} \quad W = I_M^K((\lambda/\lambda^\tau) \chi_M), \quad W' = \chi \boxtimes \chi_{\epsilon_K}.
\]  

(A.3.5)

Since \( \epsilon_K \) is a base change from \( F, \theta \) fixes it, and consequently, \( W' \) descends to a cuspidal element of \( A_2(F) \) iff \( \chi \) is not \( \theta \)-invariant. By Theorem A, \( \Pi_K \) is of type \((2, 1, 1)\) or \((1, 1, 1, 1)\) depending on whether or not \( \pi \) and \( \pi^\theta \) are induced only from \( M \), or also from another quadratic extension \( M'/K \).

In the former case, \( \lambda/\lambda^\tau \) is \( \tau \)-invariant and so \( W \) is cuspidal, which implies that \( \theta \) preserves \( W \) and \( W' \), hence \( \Pi \) is of isobaric type \((2, 1, 1)\), resp. \((2, 2)\), if \( \chi \) is, resp. is not, \( \theta \)-invariant. So we may suppose we are in the latter case, when \( \pi \) is induced from three quadratic extensions \( M, M', M'' \) say, with \( M' \) being associated to the restriction \( \xi \) of the quadratic \( \tau \)-invariant character \( \lambda/\lambda^\tau \) to \( C_K \), and with \( M'' \) associated to \( \xi \epsilon_{\epsilon_K} \). We obtain

\[
\Pi_K \simeq (\text{Ad}(\pi) \otimes \chi) \boxplus \chi \boxtimes \xi \chi \boxtimes \chi_{\epsilon_K} \boxplus \chi \boxtimes \chi_{\epsilon_K}.
\]  

(A.3.6)

Since \( \pi \) admits self-twists under \( \epsilon_{\epsilon_K}, \xi, \text{ and } \xi \epsilon_{\epsilon_K} \), we have \( \pi^\theta \simeq \pi^\vee \otimes \psi \), for any \( \psi \) in the set \( \Sigma \) of the four characters occurring on the right of (A.3.6). Suppose \( M'/F \) is Galois. In this case \( \xi \) is fixed by \( \theta \), and \( \Pi \) is of isobaric type \((1, 1, 1, 1)\), resp. \((2, 2)\), if \( \chi \) is, resp. is not, \( \theta \)-invariant. It remains to consider when \( M'/F \) is non-Galois, so that \( \theta \) sends \( \xi \) to \( \xi \epsilon_{\epsilon_K} \). Then \( \Pi \) admits an isobaric summand in \( A_1(F) \) iff \( \psi \) is \( \theta \)-invariant for some \( \psi \in \Sigma \), in which case \( \psi \) and \( \psi \epsilon_{\epsilon_K} \) are the only characters in \( \Sigma \) fixed by \( \theta \). Hence \( \Pi \) is of type \((2, 1, 1)\) or \((2, 2)\) when \( M'/F \) is non-Galois, depending on whether or not there is some \( \psi \in \Sigma \) fixed by \( \theta \).

Finally part (d) follows in the non-dihedral case from (c) since we showed in the course of the proof of (a) that \( \Pi \) is of type \((3, 1)\) when it is non-cuspidal. When \( \pi \) is dihedral, (d) is immediate from (c). \( \Box \)

A.4. The Asai case: The second proof of Theorem B

We will focus here on the proof of only the key part of (b), namely that given (\( \pi, K/F, \theta \)) as in Theorem B, \( \Pi = A_{K/F}(\pi) \) is non-cuspidal iff we have either (i) \( \pi \) and \( \pi^\theta \) are twist equivalent, or (ii) \( \pi \) is a dihedral representation induced from a biquadratic extension \( M \) of \( F \) containing \( K \). Since it is well known for non-dihedral \( \pi \) (see [A-9-A-10]) that the non-cuspidality is in fact equivalent to (i), we may assume that \( \pi \) is dihedral. We will work totally on the Weil group side and treat \( \pi \) as a representation of \( W_K \), with cuspidality corresponding to irreducibility; we will also say that \( \pi \) has CM by \( K \). Then \( \Pi \) identifies with the tensor induction \( A_{\pi}(\pi) \) of \( \pi \) to a 4-dimensional representation of \( W_F \), with the restriction \( \Pi_K \) to \( W_K \) being \( \pi \otimes \pi^\theta \). Of course \( A_{\pi}(\pi) \) makes sense for any representation \( \tau \) of \( \pi \), not necessarily two-dimensional, in particular, for \( \pi = \pi \otimes \pi^\theta \). If \( M/k \) is any quadratic extension, let \( \delta_{M/K} \) be the corresponding order 2 character of \( W_K \).

In this section we will repeatedly use the following well-known lemma from class field theory. Recall that given a subgroup \( H \) of finite index in a group \( G \), there is the notion of the transfer map from \( G/[G, G] \) to \( H/[H, H] \) which allows one to transfer characters of \( H \) to characters of \( G \). If \( K \) is a finite extension of a local (resp. global) field \( F \), then the transfer map from the characters of \( W_K \) to the characters of \( W_F \) corresponds to the restriction of the associated characters of \( K^\ast \) (resp. \( K^\ast ) \) to those of \( F^\ast \) (resp. \( F^\ast ) \).

**Lemma E.** Let \( M \) be a quadratic extension of \( K \) which is in turn a quadratic extension of \( F \), which is either a local or a global field. Then \( M \) is Galois over \( F \) if and only if \( \delta_{M/K} \) is invariant under the Galois group of \( K \).
over \( F \). If \( \delta_{M/K} \) is invariant under the Galois group of \( K \) over \( F \), then its transfer to \( W_F \) is trivial if and only if \( M \) is bi-quadratic over \( F \). If the Galois group of \( M \) over \( F \) is \( \mathbb{Z}/4 \), then the transfer of \( \delta_{M/K} \) to \( W_F \) is \( \delta_K/F \). If \( M \) is not Galois over \( F \), then the transfer of \( \delta_{M/K} \) to \( W_F \) is non-trivial, and not \( \delta_K/F \).

Recall that a finite dimensional semi-simple representation \( \tau \) of \( W_F \) is irreducible iff \( \tau \otimes \tau^\vee \) contains the trivial representation exactly once. We apply this to \( \tau = \text{As}(\pi) \). Since

\[
\text{As}(\pi) \otimes \text{As}(\pi)^\vee = \text{As}(\pi \otimes \pi^\vee),
\]

we need to analyze \( \text{As}(\pi \otimes \pi^\vee) \).

Assume that \( \pi \) is CM by three distinct quadratic extensions of \( K \), with quadratic characters \( \alpha, \beta, \alpha \beta \). In this case,

\[
\pi \otimes \pi^\vee = 1 \oplus \alpha \oplus \beta \oplus \alpha \beta,
\]

and therefore,

\[
\text{As}(\pi \otimes \pi^\vee) = \text{As}(1 \oplus \alpha \oplus \beta \oplus \alpha \beta).
\]

It is easy to see that for any two representations \( V_1 \) and \( V_2 \) of \( W_K \),

\[
\text{As}(V_1 \oplus V_2) = \text{As}(V_1) \oplus \text{As}(V_2) \oplus \text{Ind}_K^F(V_1 \otimes V_2^{\sigma_2}).
\]

Using this, and recalling that the Asai lift of a character of \( W_K \) is just its transfer to \( W_F \), we can calculate \( \text{As}(\pi \otimes \pi^\vee) = \text{As}(1 \oplus \alpha \oplus \beta \oplus \alpha \beta) \), and find that \( \text{As}(\pi \otimes \pi^\vee) \) contains more than 1 copy of the trivial representation if and only if either of the following happens:

1. The transfer of one of the characters \( \alpha, \beta, \alpha \beta \) of \( W_K \) to \( W_F \) is trivial.
2. For two distinct characters \( \gamma, \eta \) among \( \alpha, \beta, \alpha \beta \),

\[
\text{Ind}_K^F(\gamma \eta^\sigma),
\]

contains the trivial representation.

In case (1), \( \pi \) has a self-twist by a character, say \( \alpha \), such that the corresponding quadratic extension \( M_\alpha \) of \( K \) is bi-quadratic over \( F \) by Lemma E.

In case (2), we get that \( \gamma \eta^\sigma \) is the trivial character. Therefore if none of the characters \( \alpha, \beta, \alpha \beta \), are invariant under \( \text{Gal}(K/F) \), then there are two characters among this which are Galois conjugate, say \( \beta = \alpha \theta \). In this case \( \alpha \beta = \alpha \alpha \theta \), so we have a non-trivial twist of \( \pi \) which is Galois invariant. Since this twisting character is \( \alpha \alpha \theta \) with \( \alpha \) quadratic, it must be trivial on \( F^\times \), and hence gives rise to a bi-quadratic extension of \( F \) by Lemma E.

If \( \pi \) has CM by a unique quadratic extension \( L \) of \( K \), then if \( \text{As}(\pi) \) is non-cuspidal, we see by Theorem A that \( \pi \) and \( \pi^\sigma \) must both be induced from \( L \), forcing it to be Galois over \( F \). To end the proof of (part (b) of) Theorem B, it remains to prove that in this case, \( \text{Gal}(L/F) = \mathbb{Z}/2 \times \mathbb{Z}/2 \), or rather that the Galois group cannot be \( \mathbb{Z}/4 \), since it is quite straightforward to see that if \( \text{Gal}(L/F) = \mathbb{Z}/2 \times \mathbb{Z}/2 \), then indeed \( \text{As}(\pi) \) is not cuspidal.

Assume in the rest of the proof that \( L \) is a quadratic extension of \( K \) which is Galois over \( F \) with \( \text{Gal}(L/F) = \mathbb{Z}/4 = \langle \sigma \rangle \) (so that \( \theta = \sigma^2 \)), and that \( \pi = \text{Ind}_K^F(\chi) \) arises from no other quadratic extension of \( K \), with the further property that \( \pi^\sigma \neq \pi \otimes \mu \) for any character \( \mu \) of \( W_K \). The first condition translates into the condition that the character \( \chi/\chi^{\sigma^2} \) is not of order 2, and the second implies that the character \( \chi^{\sigma^2} \) is not invariant under \( \sigma^2 \).
Since $\pi \otimes \pi^\sigma = \text{Ind}_K^L(\chi \chi^\sigma) \oplus \text{Ind}_K^L(\chi^2 \chi^\sigma)$, we find that it is a sum of two distinct irreducible representations permuted by $\sigma$, therefore $\text{As}(\pi)$ must be irreducible.

References to Appendix A


References


