SYMPLECTIC LOCAL ROOT NUMBERS, CENTRAL CRITICAL $L$-VALUES, AND RESTRICTION PROBLEMS IN THE REPRESENTATION THEORY OF CLASSICAL GROUPS

WEE TECK GAN, BENEDICT H. GROSS AND DIPENDRA PRASAD

Abstract. We give a conjectural description of the restriction of an irreducible representation of a unitary group $U(n)$ to a subgroup $U(n - 1)$ over a local or global field. We formulate analogous conjectures for the restriction problem from $U(n)$ to a subgroup $U(m)$ ($m < n$) using Bessel and Fourier-Jacobi models, and also similar restriction problems for symplectic groups. The conjectures are analogs of those in [GP1] and [GP2] for the orthogonal groups. We verify these conjectures in certain low rank cases and for depth zero supercuspidal representations, and prove that the conjectures about Bessel and Fourier-Jacobi models follow from the conjectural description of the restriction of an irreducible representation of a unitary group $U(n)$ (resp. $SO(n)$) to a subgroup $U(n - 1)$ (resp. $SO(n - 1)$).

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### Introduction

In this paper, we will consider several problems about the restriction of irreducible complex representations of unitary, orthogonal, and symplectic groups to subgroups of similar type (possibly fattened by a unipotent group when there is room left). The basic question is to determine the irreducible representations of the subgroup which occur together with their multiplicities. We formulate the general problem as follows. Let \(\pi_1\) be an irreducible representation of a group \(G_1\), and let \(\pi_2\) be an irreducible representation of a subgroup \(G_2 \subset G_1\). Then \(\pi = \pi_1 \otimes \pi_2\) is an irreducible representation of the product group \(G = G_1 \times G_2\), which contains the subgroup \(H = G_2\) embedded diagonally. We say that the dual representation \(\pi_2^\vee\) occurs in the restriction of \(\pi_1\) with multiplicity \(d\) if the complex vector space \(\text{Hom}_H(\pi, \mathbb{C})\) of \(H\)-invariant linear forms on \(\pi\) has dimension \(d\).

We first consider the case when \(k\) is a local field, and \(K\) is an étale quadratic \(k\)-algebra, and \(G_1 = U(W)\) is the unitary group of a non-degenerate Hermitian space \(W\) of rank \(n\) over \(K\). We let \(G_2 \subset G_1\) be the subgroup fixing a non-isotropic line in \(W\), which is isomorphic to the unitary group \(U(W_0)\) of the orthogonal complement, of dimension \((n-1)\). Thus we are considering a generalization of the classical problem of restricting irreducible representations of the compact Lie group \(U(n)\) to the subgroup \(U(n - 1)\).

It has recently been shown by Aizenbud-Gourevitch-Rallis-Schiffmann in [AGRS], the long-awaited result (for \(k\) non-Archimedean) that for any irreducible, complex representation \(\pi = \pi_1 \otimes \pi_2\) of \(G = U(W) \times U(W_0)\), the vector space \(\text{Hom}_H(\pi, \mathbb{C})\) of \(H = U(W_0)\)-invariant linear forms on \(\pi\) has dimension \(d \leq 1\). The problem, then, is to determine for which pairs \((\pi_1, \pi_2)\), it is non-trivial.

We propose a general conjecture which answers this problem in two important cases of number theoretic interest:
• when \( k \) is local and the representation \( \pi \) of \( G(k) \) lies in a generic \( L \)-packet, and
• when \( k \) is global with ring of ad\'eles \( \mathbb{A} \) and \( \pi \) is an automorphic tempered representation of \( G(\mathbb{A}) \).

Our method assumes the Langlands parameterization of irreducible representations \( \pi \) of \( G \) into finite \( L \)-packets. (For a general discussion of the local Langlands conjecture, see [GR]. For unitary groups, where it is almost established, see [M].) We conjecture that there is a unique representation \( \pi' \) of a pure inner form \( G' = U(W') \times U(W_0') \) of \( G \) in each generic \( L \)-packet, such that \( \text{Hom}_{H'}(\pi', \mathbb{C}) \) is non-zero.

The irreducible representations in an \( L \)-packet should correspond to irreducible representations of the component group \( A_\varphi \) of the centralizer of the Langlands parameter \( \varphi \).

When \( G = U(W) \times U(W_0) \), the component group is an elementary abelian 2-group. We construct a homomorphism

\[
\chi : A_\varphi \rightarrow \langle \pm 1 \rangle
\]

using the local root numbers of symplectic summands of a natural symplectic representation \( V \) of the \( L \)-group. We conjecture that the unique representation \( \pi' \) in the \( L \)-packet with an \( H' \)-invariant linear form corresponds to the character \( \chi \). This is similar to our conjectures [GP1] in the orthogonal case; in the unitary case, \( V \) has dimension \( 2n \cdot (n - 1) \) and is induced from a tensor product of standard representations.

As in the orthogonal case [GP2], one can define the character \( \chi \) and formulate a more general conjecture on restriction from \( U(W) \) to \( U(W_0) \), where \( W_0 \) is a non-degenerate subspace of odd codimension in \( W \). This conjecture also includes a generic character of a unipotent subgroup of \( U(W) \), when the codimension of \( W_0 \) in \( W \) is greater than one, and corresponds to the classical theory of Bessel models. We show that the general conjecture about Bessel models follows from the original one (when \( W_0 \) has codimension 1 in which case the subgroup \( H \) is reductive, unlike higher codimension case when the subgroup has a unipotent part) in both unitary and orthogonal cases.

We then turn to the restriction problem for Hermitian spaces \( W_0 \subset W \) over a global field \( k \), with quadratic étale algebra \( K \). Let \( G = U(W) \times U(W_0) \) and assume that \( \pi \) is an irreducible automorphic representation of \( G(\mathbb{A}) \), where \( \mathbb{A} \) is the ring of ad\'eles of \( k \). If the vector space \( \text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \) is non-zero, our local conjecture predicts that the global root number \( \epsilon(\pi, V, 1/2) \) is equal to 1. If we assume \( \pi \) to be tempered, then our calculation of global root numbers, and general conjectures of Langlands and Arthur predict that \( \pi \) appears with multiplicity one in the discrete spectrum of \( G \). We conjecture that the period integrals on the corresponding space of functions

\[
f \mapsto \int_{H(k) \backslash H(\mathbb{A})} f(h) \, dh
\]

gives a non-zero element in \( \text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \) if and only if the central critical \( L \)-value \( L(\pi, V, 1/2) \) is non-zero.
One case in which all of these conjectures are known to be true is when the quadratic étale algebra $K$ is split: $K \cong k \times k$. Then $G_1 \cong \text{GL}_n(k)$ and $G_2 \cong \text{GL}_{n-1}(k)$. When $k$ is local, and $\pi$ is a generic representation of $G = G_1 \times G_2$, the theory of Rankin-Selberg integral, cf. [P4, Theorem 3], shows that the complex vector space $\text{Hom}_\mathbb{H}(\pi, \mathbb{C})$ has dimension $1$ (assuming [AGRS] which shows that the dimension is at most $1$). This agrees with our local conjecture, as the $L$-packets for $G = \text{GL}_n(k) \times \text{GL}_{n-1}(k)$ contain single element. If $k$ is global and $\pi$ is a tempered automorphic representation of $G(A)$, then $\pi$ appears with multiplicity one in the discrete spectrum and the periods over $H(k) \backslash H(\mathbb{A})$ give a non-zero linear form if and only if the tensor product $L$-function $L(\pi, \text{std}_n \otimes \text{std}_{n-1}, s)$ is non-zero at $s = \frac{1}{2}$ [JPSS]. In the split case, $V$ is the sum of $\text{std}_n \otimes \text{std}_{n-1}$ and its dual, and the local and global root numbers are all equal to 1.

We also present some material on the restriction of representations of the symplectic and metaplectic groups, using the Weil representations and the Fourier-Jacobi models. All of the situations studied involve representations of the Weil-Deligne group of the form $V = V_1 \otimes V_2$ with $V_1$ symplectic and $V_2$ orthogonal. We recall that in [GP1] and [GP2], the orthogonal representation $V_2$ was always even dimensional, being the $L$-group of an even dimensional orthogonal group. It is thus natural to look for representation theoretic contexts which might involve representations of the Weil-Deligne group of the form $V_1 \otimes V_2$ with $V_1$ symplectic and $V_2$ orthogonal of odd dimension. This suggests looking for branching laws involving odd orthogonal and symplectic groups. This does not seem to lead to any meaningful representation theoretic question. However, it is known that $\text{Sp}_{2n}(\mathbb{C})$ can be taken to be the $L$-group also of the metaplectic group $\text{Mp}_{2n}$, and then a possible branching law involves restriction from Symplectic to metaplectic group, or vice-versa. This is how the Weil representation, and the Fourier-Jacobi models for symplectic group arise in this paper, completing the picture of [GP1] and [GP2]. In fact, methods of theta correspondence allows one to reduce these questions once again to the basic branching laws of [GP1] involving $(\text{SO}_n, \text{SO}_{n-1})$ as we see in this paper.

Finally, we provide evidence for our conjectures in various cases:

- unitary groups of low rank, namely $U(2) \times U(1)$ and $U(3) \times U(2)$;
- discrete series representations in the real case;
- tame parameters in the $p$-adic case.

As already mentioned, the last two authors proposed a similar conjecture in [GP1] for the restriction of irreducible representations of $\text{SO}(n, k)$ to $\text{SO}(n-1, k)$. Since then, there has been considerable progress in the local case [GR, GW] as well as in the global case [GJR], but the conjecture remains open in general. Shortly after the publication of [GP1], it was also realized that the conjectural framework for $(\text{SO}(n), \text{SO}(n-1))$ should continue to hold for $(U(n)), U(n-1))$, but as the case of unitary groups seemed so similar to that of the orthogonal groups, the last two authors did not write it down. However, given the continued interest in these restriction problems, the recent proof of the multiplicity one result [AGRS], the recent progress in the local Langlands conjecture for unitary groups [M] and the fact that the unitary groups $U(n)$ are in many ways more accessible (being...
related to $GL(n)$ by base change and supporting Shimura varieties), it seems an appropriate moment to give an account of the Gross-Prasad conjecture for unitary groups. In fact, as we were just finishing writing the paper, we learnt of an announcement by Waldspurger of a multiplicity 1 theorem (and not just less than or equal to one) for certain tempered $L$-packets for $(SO(n), SO(n-1))$, assuming certain natural statements about characters in an $L$-packet.

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Part 1. **CONJECTURES**

1. **HERMITIAN SPACES AND UNITARY GROUPS**

Let $k$ be a field, $K$ a separable quadratic field extension of $k$ and $\tau$ the non-trivial involution of $K$ fixing $k$.

A Hermitian space $M$ of rank $n$ is an $n$-dimensional $K$-vector space, equipped with a non-degenerate Hermitian form

$$\beta : M \times M \rightarrow K.$$  

The Hermitian condition is that $\beta$ is $K$-linear in the first variable and satisfies

$$\beta(v, w)^\tau = \beta(w, v) \quad \text{for any } v, v \in M.$$  

The non-degenerate condition is that the map $M \rightarrow \text{Hom}_K(M, K)$ taking a vector $w$ in $M$ to the linear form $v \mapsto \beta(v, w)$ is a $K$-linear bijection.

We say that the Hermitian space $M$ is **split** if it contains an isotropic subspace $H$ (i.e., a subspace on which the restriction of $\beta$ is trivial) of the largest possible dimension, namely:

$$\begin{cases} n/2, & \text{if } n \text{ is even;} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$  

When $n$ is even, there is a unique split Hermitian space of rank $n$ up to isomorphism; it is given by $M = H \oplus H^\vee$ where $H^\vee$ is isotropic and dual to $H$ under $\beta$. When $n$ is odd, a split Hermitian space of rank $n$ has the form $M = H \oplus H^\vee \oplus \langle \alpha \rangle$, where $\langle \alpha \rangle$ is the rank 1 Hermitian space determined by $\alpha \in k^\times$. Thus, there are $\# [k^\times / N_{K/k}(K^\times)]$ split Hermitian spaces of odd rank $n$.

The unitary group $U(M)$ is the subgroup of $GL_K(M)$ preserving $\beta$:

$$U(M) = \{ T \in GL_K(M) \mid \beta(Tv, Tw) = \beta(v, w) \}.$$  

It is a reductive group over $k$ of dimension $n^2$ and is quasi-split precisely when $M$ is split. The group $U(1) = U(\wedge^n M)$ is a 1-dimensional torus whose isomorphism class depends only on $K/k$ and not on $M$. This torus is isomorphic to the center of $U(M)$ and to the abelianization of $U(M)$. We normalize the maps

$$U(1) \xrightarrow{\text{center}} U(M) \xrightarrow{\det} U(1)$$

so that the composite is multiplication by $n$.

We can now formulate one of the main questions we will study in this paper. Let $M$ be a split Hermitian space of dimension $n \geq 1$, and let $M_0 \subset M$ be a non-degenerate split Hermitian subspace of codimension 1 in $M$. Writing

$$M = M_0 \oplus M_0^\perp \quad \text{with} \quad \dim M_0^\perp = 1,$$

we obtain a homomorphism of algebraic groups over $k$:

$$i : U(M_0) \longrightarrow U(M)$$

mapping a unitary transformation of $M_0$ to one which acts by the identity on the line $M_0^\perp$ in $M$. This gives a homomorphism

$$j = i \times 1 : U(M_0) \longrightarrow U(M_0) \times U(M_0^\perp).$$

Our aim is to study the restriction of irreducible complex representations $\pi \otimes \pi_0$ of $U(M) \times U(M_0)$ to the subgroup $U(M_0)$ when $k$ is a local field. More precisely, we want to determine exactly when $\text{Hom}_{U(M_0)}(\pi \otimes \pi_0, \mathbb{C})$ is nonzero. To do this, we will need to recall the conjectural Langlands parameterization.

### 2. Self-Dual Representations

Let $\hat{G}$ denote a complex reductive group. We fix a pinning (épinglage) of $\hat{G}$ and let

$$\hat{T} \subset \hat{B} \subset \hat{G}$$

be the corresponding maximal torus and Borel subgroup. Consider the semi-direct product

$$\hat{G} \rtimes \langle 1, \tau \rangle,$$

where the involution $\tau$ acts on $\hat{G}$ via a pinned automorphism (possibly trivial) which maps to the opposite involution in $\text{Out}(\hat{G})$. Then for every complex algebraic representation $V$ of $\hat{G}$, the conjugate representation $V^\tau$ is isomorphic to the dual representation $V^\vee$. In particular, the induced representation

$$\text{Ind}(V) = \text{Ind}(V^\tau) = \text{Ind}(V^\vee)$$

of $\hat{G} \rtimes \langle 1, \tau \rangle$ is self-dual. For example, when $V = \mathbb{C}$ is the trivial representation, then $\text{Ind}(\mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}(\chi)$ where $\chi$ is the non-trivial character of $\hat{G} \rtimes \langle 1, \tau \rangle$ with trivial restriction to $\hat{G}$. 
The pinning determines a principal homomorphism
\[ \varphi : SL_2 \longrightarrow \hat{G} \]
which is fixed by \( \tau \). Hence, we obtain a homomorphism
\[ \varphi \times 1 : SL_2 \times \langle 1, \tau \rangle \longrightarrow \hat{G} \rtimes \langle 1, \tau \rangle. \]
Let \( \epsilon = \varphi(-1) \) in \( \hat{G} \). Then \( \epsilon \) is in \( \hat{T} \), \( \epsilon^2 = 1 \) and \( \epsilon \) acts trivially on each root space of \( \text{Lie}(\hat{G}) \). Hence \( \epsilon \) lies in the center \( Z(\hat{G}) \). Since \( \epsilon \) is fixed by \( \tau \), it lies in the center \( Z(\hat{G})^\tau \) of \( \hat{G} \rtimes \langle 1, \tau \rangle \).

The following result is due to Deligne and generalizes a result in Bourbaki [Bo].

**Proposition 2.1.** Every representation \( W \) of \( \hat{G} \rtimes \langle 1, \tau \rangle \) is self-dual. If \( W \) is irreducible, the invariant pairing on \( W \) is \((\epsilon|W)\)-symmetric.

The proof is similar to that in Bourbaki. To determine the sign of the pairing on an irreducible \( W \), one restricts \( W \) to \( SL_2 \times \langle 1, \tau \rangle \) and observes that there is an irreducible component of multiplicity one.

The groups \( \hat{G} \rtimes \langle 1, \tau \rangle \) which arise in this section are precisely the \( L \)-groups (in the sense of Langlands) of anisotropic groups \( G \) over \( \mathbb{R} \), with \( \langle 1, \tau \rangle = \text{Gal}(\mathbb{C}/\mathbb{R}) \). As a special case, we have the \( L \)-group of the compact Lie group \( G = U(n) \) associated to a positive definite Hermitian space of rank \( n \). Here
\[ \hat{G} \cong \text{GL}_n(\mathbb{C}) \quad \text{and} \quad \tau(A) = J \cdot tA^{-1} \cdot J^{-1} \]
with
\[ J = \begin{pmatrix}
1 & \cdots & \cdots & \cdots \\
\vdots & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
\vdots & \vdots & \vdots & -1
\end{pmatrix}. \]

In this case, the center of the \( L \)-group is
\[ (\mathbb{C}^\times)^\tau = \langle \pm 1 \rangle \]
and
\[ \epsilon = J^2 = (-1)^{n-1}. \]
Indeed, the principal \( SL_2 \longrightarrow \text{GL}_n \) is given by the representation \( \text{Sym}^{n-1} \). The transfer homomorphism
\[ \text{Ver} : \langle L \hat{G} \rangle^{ab} = \langle 1, \tau \rangle \longrightarrow \text{GL}_n(\mathbb{C})^{ab} = \mathbb{C}^\times \]
is trivial, as $L^G$ is a semi-direct product.

**Proposition 2.2.** Let $W$ be the irreducible faithful representation of dimension $2n$ of 

$$Lu(n) = GL_n(\mathbb{C}) \rtimes (1, \tau)$$

which is induced from the standard representation $V = \mathbb{C}^n$ of $GL_n(\mathbb{C})$. Then $W = \text{Ind}(V) = \text{Ind}(V^\vee)$ is symplectic when $n$ is even and orthogonal with $\det = \chi$ when $n$ is odd.

**Proof.** This follows from our calculation of $\epsilon$ and the fact that $\det(\text{Ind}(V)) = \text{Ver}(\det V) \cdot \chi^\text{dim}V$. \qed

Next, consider the compact Lie group $G = U(n) \times U(n-1)$. The $L$-group $L^G$ is isomorphic to $(GL(W) \times GL(V)) \rtimes (1, \tau)$, with $\dim W = n - 1$ and $\dim V = n$. This $L$-group has two faithful irreducible representations of dimension $2 \cdot n \cdot (n - 1)$:

$$\begin{align*}
U &= \text{Ind}(W \otimes V) = \text{Ind}(W^\vee \otimes V^\vee), \\
U' &= \text{Ind}(W \otimes V^\vee) = \text{Ind}(W^\vee \otimes V).
\end{align*}$$

They are both symplectic, as the principal $SL_2$ acts on the tensor products by the representation $\text{Sym}^{n-2} \otimes \text{Sym}^{n-1}$. We have the decomposition

$$U \oplus U' = \text{Ind}(W) \otimes \text{Ind}(V).$$

3. Langlands Parameters for Unitary Groups

We now assume that $K/k$ are locally compact fields. Let $M$ be a split Hermitian space of rank $n$ over $k$. The $L$-group of $G$ is isomorphic to

$$L^G \cong GL(V) \rtimes \text{Gal}(K/k),$$

where $V$ is a complex vector space of dimension $n$.

Let $W'_k$ denote the Weil group $W_k$ of $k$ if $k = \mathbb{R}$, and the product $W_k \times SL_2(\mathbb{C})$ if $k$ is non-Archimedean. A Langlands parameter for $G = U(M)$ is a homomorphism

$$\varphi : W'_k \longrightarrow L^G$$

satisfying some conditions. In particular, the restriction $\varphi_K$ of $\varphi$ to the subgroup $W'_K$ of index 2 is a complex representation

$$\varphi_K : W'_K \longrightarrow GL(V).$$

Let $s \in W'_k$ be a representative of the non-trivial coset of $W'_K$. Then

$$\sigma \mapsto s \sigma s^{-1}$$
gives an outer automorphism of $W'_K$ and the representation $V$ given by $\varphi_K$ satisfies

$$V^s \cong V^\vee.$$ 

Let $V = \oplus_i m_i V_i$ be the decomposition of the representation $\varphi_K$ into irreducible representations $V_i$ of $W'_K$, occurring with multiplicities $m_i$. The centralizer of the image of $\varphi_K$ in $\text{GL}(V)$ is then isomorphic to the product $\prod_i \text{GL}_{m_i}(\mathbb{C})$ by Schur's lemma. Our aim in this section is to determine the centralizer $C_\varphi$ in $\text{GL}(V)$ of the image of $\varphi$, as a subgroup of $\prod_i \text{GL}_{m_i}(\mathbb{C})$, and to examine the group $A_\varphi$ of connected components of $C_\varphi$.

Since $V^s \cong V^\vee$, there are two possibilities for each irreducible factor $V_i$:

$$\begin{cases} V_i^s \cong V_j^\vee \text{ with } j \neq i; \\ V_i^s \cong V_i^\vee. \end{cases}$$

The first case implies that $m_i = m_j$. The second implies that there is a non-degenerate bilinear form (unique up to scaling)

$$B : V_i \times V_i \longrightarrow \mathbb{C}$$

which satisfies:

$$B(\sigma v, s \sigma s^{-1} w) = B(v, w).$$

Since the form

$$B'(v, w) := B(w, s^2 v)$$

enjoys the same properties, we have $B = c_i \cdot B'$ for some $c_i \in \mathbb{C}^\times$. We find that $c_i^2 = 1$ and the induced representation $\text{Ind}(V_i)$ of $W'_K$ is $c_i$-symmetric. One could define $c_i = \pm 1$ using the sign of the self-duality of $\text{Ind}(V_i)$ if that representation were irreducible. However, when $V_i = V_i^s = V_i^\vee$ is both conjugate self-dual and self-dual, $\text{Ind}(V_i)$ is reducible and has a pairing of either of the signs.

**Proposition 3.1.** The centralizer in $\text{GL}(V)$ of the image of $\varphi$ is determined by the irreducible decomposition $V = \oplus m_i V_i$ of the representation $\varphi_K$ and the signs $c_i$ of the irreducible conjugate self-dual factors $V_i$ as follows:

1. If $V_i^s \cong V_j^\vee$ with $i \neq j$, then $m_i = m_j$ and the centralizer of $\varphi$ is the diagonal subgroup $\text{GL}_{m_i}(\mathbb{C}) \hookrightarrow \text{GL}_{m_i}(\mathbb{C}) \times \text{GL}_{m_i}(\mathbb{C})$.
2. If $V_i^s \cong V_i^\vee$ and $c_i = (-1)^n$, then $m_i$ is even and the centralizer of $\varphi$ is the symplectic subgroup $\text{Sp}_{m_i}(\mathbb{C}) \subset \text{GL}_{m_i}(\mathbb{C})$.
3. If $V_i^s \cong V_i^\vee$ and $c_i = (-1)^{n-1}$, then the centralizer of $\varphi$ is the orthogonal subgroup $\text{O}_{m_i}(\mathbb{C}) \subset \text{GL}_{m_i}(\mathbb{C})$.

**Proof.** This is proved in [P1].
As a consequence, the full centralizer of \( \varphi \) in \( \text{GL}(V) \) is a product

\[ C_\varphi = \prod_{(i,i') \text{ of type 1}} \text{GL}_{m_i}(\mathbb{C}) \times \prod_{j \text{ of type 2}} \text{Sp}_{m_j}(\mathbb{C}) \times \prod_{k \text{ of type 3}} \text{O}_{m_k}(\mathbb{C}) \]

over the factors of the 3 types in the proposition above. Observe that we have the congruence

\[ n = \dim V = \sum_i m_i \dim V_i \equiv \sum_{i \text{ of type 3}} m_i \dim V_i \mod 2. \]

From the above discussion, it is easy to determine the component group \( A_\varphi = \pi_0(C_\varphi) \) of the centralizer of \( \varphi \) in \( \text{GL}(V) \):

Corollary 3.2. Let

\[ \varphi : W'_k \to \text{GL}(V) \times \text{Gal}(K/k) \]

be a Langlands parameter for \( U(M) \) where \( M \) is a split Hermitian space of rank \( n \). Then we have:

1. The parameter \( \varphi \) is a discrete parameter, i.e. \( C^0_\varphi \) is trivial, if and only if only representations of type 3 occur, and all the multiplicities \( m_i \) are 1.
2. \( A_\varphi \) is an elementary abelian 2-group of rank \( \leq n \);
3. \( A_\varphi \) has a distinguished basis over \( \mathbb{Z}/2\mathbb{Z} \), indexed by the irreducible factors \( V_i \) of the representation \( \varphi_K \) which are of type 3 in the sense of proposition 3.1. For each \( i \) of type 3, the corresponding basis element is given by the image \( \epsilon_i \equiv g_i \) of an element \( g_i \in O_{m_i}(\mathbb{C}) \) with \( \det(g_i) = -1 \).

Remark: We have noted that the \( L \)-group \( L\text{U}(n) \) has an irreducible representation of dimension \( 2n \) which is either orthogonal or symplectic. Thus given a parameter \( \varphi : W'_k \to L\text{U}(n) \) for a unitary group, we get a parameter \( \varphi' : W'_k \to \text{O}(2n, \mathbb{C}) \), or \( \varphi' : W'_k \to \text{Sp}(2n, \mathbb{C}) \) depending on whether \( n \) is odd or even. This induces a mapping on the group of connected components: \( A_\varphi \to A_{\varphi'} \). But because of the possibility of an irreducible representation \( \varphi : W_k \to \text{GL}(m, \mathbb{C}) \) such that \( \varphi \not\cong \varphi^\vee \), but \( \varphi \cong \varphi^\vee \otimes \omega_{K/k} \), the mapping on the component groups is not injective (such a parameter restricted to \( W_K \) remains irreducible); and because of the possibility of an irreducible representation \( \varphi : W_K \to \text{GL}(m, \mathbb{C}) \) such that \( \varphi \cong \varphi^\vee \), but \( \varphi \not\cong \varphi^\ast \), the mapping on the component groups is not surjective (such a parameter induced to \( W_k \) is self-dual and irreducible). We note, however, that for \( K/k = \mathbb{C}/\mathbb{R} \), there are no such representations, and therefore the mapping of the component groups is an isomorphism in this case.

Remark: Since at some places later, we will consider the parameter \( \varphi \) of a unitary group only through its restriction to \( K \), denoted here by \( \varphi_K \), it is nice to note, cf. [BC, proposition A.11.3] that a conjugate-self-dual representation \( \varphi_K \) of \( W'_K \) extends to a parameter of the unitary group in a unique way in either of the two situations:
• $\varphi_K$ is a sum of irreducible representations, each occurring with multiplicity 1.

• $\varphi_K$ is an abelian representation.

Now one has a natural inclusion

$$Z(L^1 G) = \langle \pm 1 \rangle \hookrightarrow C_\varphi$$

which induces a homomorphism of groups

$$Z(L^1 G) \longrightarrow A_\varphi.$$

The following lemma determines the image of this homomorphism:

**Lemma 3.3.** The image of $-1$ is the element $\sum_i m_i \epsilon_i$, where the summation is over the representations $V_i$ of type 3, and $m_i$ is the multiplicity of $V_i$ in $V$.

When $n$ is odd, the image of $-1$ is always non-trivial in $A_\varphi$, as

$$\sum_{i \text{ of type 3}} m_i \dim V_i \equiv n \mod 2.$$

When $n$ is even, $-1$ is trivial in $A_\varphi$ precisely when all $V_i$ of type 3 occur with even multiplicity in $V$.

We conclude this section by defining a distinguished character $\eta$ (possibly trivial when $n$ is even) of $A_\varphi$. First define $\eta$ as a homomorphism $C_\varphi \longrightarrow \langle \pm 1 \rangle$, using the formula

$$\eta(a) = (-1)^{\dim V_a = -1}.$$

It is easy to see that this descends to a character of $A_\varphi$ with

$$\eta(-1) = (-1)^n.$$

On the basis elements $\epsilon_i$, one has

$$\eta(\epsilon_i) = (-1)^{\dim V_i}.$$

4. **The Langlands Conjecture for $U(M)$**

Let $M$ be a split Hermitian space of rank $n \geq 1$ and let $G = U(M)$. In this section, we briefly recall the local Langlands conjecture for $G$ and its pure inner forms.

The pointed set $H^1(k, U(M))$ parameterizes the isomorphism classes of rank $n$ Hermitian spaces $(M', \beta')$. When $k = \mathbb{R}$, there are $n + 1$ classes, determined by the signature of the Hermitian form $\beta'(v, v)$. When $k$ is non-Archimedean,

$$H^1(k, U(M)) \xrightarrow{\det} H^1(k, U(1)) = k^\times / NK^\times$$

is a group of order 2, which Kottwitz has shown is dual to $Z(L^1 G)$. The two Hermitian spaces $M$ and $M'$ are determined by their Hermitian discriminant, and when $n$ is odd, they are both split.
Recall that $G = U(M)$ is quasi-split over $k$. Let $B$ be a Borel subgroup and $N$ its unipotent radical. A homomorphism

$$f : N \longrightarrow \mathbb{G}_a$$

of unipotent algebraic groups over $k$ is called *generic* if its stabilizer in $T = B/N$ is equal to the center $U(1)$. Fix a $T$-orbit on the set of generic characters of $N$; there are two orbits when $n$ is even and one orbit when $n$ is odd. Then, relative to this choice of $T$-orbit of generic characters of $N$, one has:

**The Langlands conjecture for $G = U(M)$:**

(i) The irreducible admissible complex representations $\pi$ of $G(k)$ can be parametrized by pairs $(\phi, \rho)$, where $\phi$ is a Langlands parameter for $G$ and $\rho$ is an irreducible representation of the finite group $A_\phi$ which is trivial on the image of $Z(^L G)$.

(ii) Two pairs $(\phi, \rho)$ and $(\phi', \rho')$ correspond to the same irreducible representation $\pi$ if and only if they are conjugate by an element of $\hat{G} = GL_n(\mathbb{C})$.

(iii) If $\phi$ is a generic parameter, in the sense that its adjoint $L$-function $L(s, \phi, Ad)$ is regular at $s = 1$, then the trivial representation $\rho = 1$ should correspond to the unique representation in the $L$-packet on which $N$ has an $f$-generic linear form.

More generally, a pair $(\phi, \rho)$ (up to $\hat{G}$-conjugacy) with $\rho$ an arbitrary representation of $A_\phi$ should parameterize an irreducible representation $\pi(\phi, \rho)$ of a pure inner form $U(M')$ of $U(M)$. When $k$ is non-Archimedean, the group $G' = U(M')$ will be determined by the restriction of $\rho$ to the image of $Z(^L G)$. The distinguished character $\eta$ of $A_\phi$ defined in the previous section should correspond to a further generic representation in the $L$-packet, which will be on a quasi-split pure inner form $U(M')$ when $n$ is odd.

By the recent work of Arthur and Moeglin [M], the local Langlands conjecture for $G = U(M)$ is now essentially known.

5. **Parameters for $U(M) \times U(M_0)$**

We now consider the Langlands parameters for the group $G = U(M) \times U(M_0)$. Let $K/k$ be a separable quadratic extension of local fields. Let $M$ be a split Hermitian space of dimension $n \geq 1$, and let $M_0 \subset M$ be a non-degenerate split Hermitian space of codimension 1. Then we have a natural homomorphism of quasi-split groups over $k$:

$$j : U(M_0) \longrightarrow U(M_0) \times U(M).$$

By the local Langlands conjecture discussed in the previous section, the irreducible representations $\pi$ of $G(k)$ and its pure inner forms are parametrized by pairs $(\phi, \rho)$ where

$$\phi : W'_k \longrightarrow ^L G = (GL(V_0) \times GL(V)) \rtimes \text{Gal}(K/k)$$
and
\[ \rho : A \rightarrow \langle \pm 1 \rangle \]
is an irreducible representation of \( A \). We have seen in Section 2 that \( L^G \) has two irreducible symplectic representations of dimension \( 2 \cdot n \cdot (n - 1) \):
\[ U = \text{Ind}(V_0 \otimes V) \quad \text{and} \quad U' = \text{Ind}(V_0 \otimes V^\vee). \]
We will need to distinguish them, which we can do using the embedding of Hermitian spaces \( M_0 \hookrightarrow M \).

Let \( M_0 = \bigoplus_{i=1}^{n-1} L_i \) be an orthogonal decomposition into non-isotropic lines, and complete this to an orthogonal decomposition of \( M \) by taking \( L_n = M_0^\perp \). These decompositions give maximal tori \( T_0 \) and \( T \) in the unitary groups. The corresponding characters of these tori on the lines \( L_i \) give basis \[ \langle e_1, \ldots, e_{n-1} \rangle \quad \text{and} \quad \langle e_1, \ldots, e_n \rangle \]
of \( X^*(T_0) \) and \( X^*(T) \) respectively. We take the dual bases
\[ \langle e_1^\vee, \ldots, e_{n-1}^\vee \rangle \quad \text{and} \quad \langle e_1^\vee, \ldots, e_n^\vee \rangle \]
in \( X^*(\hat{T}_0) \) and \( X^*(\hat{T}) \), and let \( V_0 \) and \( V \) be the irreducible representations of the dual groups with these weights. This gives an isomorphism
\[ \hat{G} \cong \text{GL}(V_0) \times \text{GL}(V) \]
and allows us to select the representation \( U = \text{Ind}(V_0 \otimes V) \) of \( L^G \).

The embedding \( M_0 \hookrightarrow M \) also selects one of the two orbits of generic \( f : N_G \rightarrow \mathbb{G}_a \). We have
\[ N_G^{ab} = N_0^{ab} + N^{ab} \]
with \( \dim N_0^{ab} = n - 2 \) and \( \dim N^{ab} = n - 1 \). Assume that \( n \) is even, so that \( \dim M_0 \) is odd and there is a unique generic orbit of \( f_0 : N_o \rightarrow \mathbb{G}_a \). Since a maximal isotropic subspace \( H \subset M \) determines a maximal isotropic subspace \( H_0 = H \cap M_0 \subset M_0 \), we find that
\[ N^{ab} = N_0^{ab} + L \]
where \( L \) is a line on which \( Z(U(M_0)) \) acts non-trivially. We define \( f : N^{ab} \rightarrow \mathbb{G}_a \) as the sum \( f_0 + l \) where \( l \) is the linear form on \( L \) given by
\[ \lambda \mapsto \frac{\beta(\lambda v, v)}{\beta(v, v)}, \quad \text{where} \quad \langle v \rangle = M_0^\perp. \]
This determines an orbit of generic characters on \( N_G \) when \( n \) is even. When \( n \) is odd, we complete \( M_0^\perp \) to a hyperbolic plane \( P \) and use the codimension one subspace \( P^\perp \subset M_0 \subset M \) to give an orbit of generic characters on \( N_G \).

Having chosen a generic \( f \) for \( N_G \), the local Langlands conjecture predicts that for generic parameters \( \varphi \) of \( G = U(M_0) \times U(M) \), the pairs \( (\varphi, \rho) \) parameterize irreducible representations of \( G(k) \) and its pure inner forms \( G'(k) \), in such a way that the trivial character \( \rho = 1 \).
corresponds to the unique $f$-generic representation in the $L$-packet associated to $\varphi$. The groups $G'$ have the form $U(M'_0) \times U(M')$, where $M'_0$ is a Hermitian space of rank $n - 1$ and $M'$ is of rank $n$. If $M'_0$ embeds as a Hermitian subspace of $M'$ and $M'/M'_0 \cong M/M_0$ as 1-dimensional Hermitian spaces, we call the pair $(M'_0, M')$ relevant.

The center of $L^G$ is now the group
$$Z(L^G) = (\mathbb{C}^\times \times \mathbb{C}^\times)^\tau = \langle \pm 1 \rangle \times \langle \pm 1 \rangle.$$

In the non-Archimedean case, a pure inner form corresponds to a character $\rho : Z(L^G) \longrightarrow \langle \pm 1 \rangle$.

The pair $(M'_0, M')$ associated to $\rho$ is relevant if and only if $\rho(-1, -1) = 1$.

In the real case, this is a necessary condition, but the relevancy also depends on the signatures of $M'_0$ and $M'$.

6. The Local Conjecture

After the preparation of the previous sections, we are now ready to return to the problem formulated in Section 1. Here is our local conjecture:

**Conjecture 6.1.** Let $\varphi$ be a generic Langlands parameter for $G = U(M_0) \times U(M)$. There is a (unique) character $\chi : A_x \longrightarrow \langle \pm 1 \rangle$ which satisfies:

1. $\chi(-1, -1) = 1$, so the pair $(M'_0, M')$ of Hermitian spaces associated to $\chi$ is relevant;
2. the irreducible representation $\pi(\varphi, \chi)$ of $G'(k) = U(M'_0) \times U(M')$, associated to $(\varphi, \chi)$ under the local Langlands correspondence, satisfies:
   $$\dim \text{Hom}_{U(M'_0)}(\pi(\varphi, \chi), \mathbb{C}) = 1;$$
3. for any other relevant $\rho \neq \chi$,
   $$\text{Hom}_{U(M'_0)}(\pi(\varphi, \rho), \mathbb{C}) = 0.$$
4. The representation $\pi(\varphi, \chi)$ of $G'(k) = U(M'_0) \times U(M')$ which has a non-zero linear form invariant under $U(M'_0)$, lives on $G'$ such that the Hermitian space in the even number of variables has the discriminant of a maximally split Hermitian space (hence $G'$ is quasi-split if $k$ is non-Archimedean) if and only if
   $$\epsilon(\frac{1}{2}, \varphi_1 \otimes \varphi_2, \psi) = 1,$$
   where the restriction of $\varphi$ to $K$, $\varphi_K$, is of the form $\varphi_K = \varphi_1 \times \varphi_2$, and the character $\psi$ of $K$ is chosen to be trivial on $k$ (the value of the epsilon factor at $1/2$ is independent of $\psi$).
In other words, the conjecture says that one has “multiplicity one for restriction in the L-packet of ϕ”:

$$\sum_{\text{relevant } \rho} \dim \text{Hom}_{U(M_0)}(\pi(\phi, \rho), \mathbb{C}) = 1.$$ 

Note again that the bijection of irreducible representations with pairs (ϕ, ρ) depends on our choice of a generic f. However, observe that for the purpose of this conjecture, one only need to have the partition of the set of irreducible representations of U(n) × U(n − 1) into L-packets; one does not really need the parameterization of these L-packets by Langlands parameters, or the fine parameterization of the members of an L-packet by the characters of the component group.

To conclude this section, we shall describe a slightly refined local conjecture which assumes that one has a parameterization of the L-packets by Langlands parameters. It does not require the precise parameterization of the members of an L-packet by the characters of the component group, which can be a very delicate issue. This conjecture is typically what is checked in practice.

**Conjecture 6.2.**

1. Fix a generic L-parameter for U(M):

$$\phi : W'_k \to GL(V) \times \text{Gal}(K/k)$$

and consider its restriction ϕ_K to the subgroup W_K of index 2. Write

$$\phi_K = n_1\sigma_1 \oplus n_2\sigma_2 \oplus \ldots \oplus n_r\sigma_r \oplus \tau,$$

where the σ_i’s are the distinct irreducible conjugate-self-dual representations of W_K appearing in ϕ_K such that the sign c(σ_i) of the bilinear form B_i on σ_i is equal to $(-1)^{\dim V + 1}$.

Fix an irreducible representation π of U(M) (a pure inner form of U(M)) in the L-packet Π(ϕ) associated to ϕ. For any generic Langlands parameter ϕ_0 of U(M_0) with associated L-packet Π(ϕ_0), set

$$\text{Hom}(\pi, \phi_0) := \bigoplus_{\pi_0 \in \Pi(\phi_0)} \text{Hom}_{U(M_0)}(\pi \otimes \pi_0, \mathbb{C}).$$

If ϕ_0 and ϕ'_0 are two generic Langlands parameters of U(M_0) such that

$$\text{Hom}(\pi, \phi_0) \neq 0 \quad \text{and} \quad \text{Hom}(\pi, \phi'_0) \neq 0,$$

then for each i, we have:

$$\epsilon(1/2, \sigma_i \otimes \phi_0, \psi) = \epsilon(1/2, \sigma_i \otimes \phi'_0, \psi) \in \pm 1,$$
where $\psi$ is a fixed non-trivial additive character on $K$ which is trivial in $k$ (the values of the epsilon factor actually depend on $\psi$). In particular, $\pi$ determines a character $\chi_\pi$ on $A_\varphi = (\mathbb{Z}/2\mathbb{Z})^r$, whose value on the non-trivial element in the $i$-th copy of $\mathbb{Z}/2\mathbb{Z}$ is equal to $\epsilon(1/2, \sigma_i \otimes \varphi_{0,K}, \psi)$ for any $\varphi_0$ such that $\Hom(\pi, \varphi_0) \neq 0$.

(2) The association $\pi \mapsto \chi_\pi$ gives a bijection between the $L$-packet $\Pi(\varphi)$ associated to $\varphi$ and the set of irreducible characters of the component group $A_\varphi$.

(3) One can exchange the role of $U(M)$ and $U(M_0)$ in the above statements. More precisely, suppose that we are given a Langlands parameter $\varphi_0$ of $U(M_0)$ such that $\varphi_{0,K} = n_1 \sigma_1 + n_2 \sigma_2 + \cdots + n_r \sigma_r + \tau$ as above and an element $\pi_0$ in the associated $L$-packet $\Pi(\varphi_0)$. Then the collection of epsilon factors $\epsilon(1/2, \varphi_K \otimes \sigma_i, \psi)$ is independent of the choice of a generic Langlands parameter $\varphi$ of $U(M)$ for which $\Hom(\varphi, \pi_0) := \bigoplus_{\pi \in \Pi(\varphi)} \Hom_{U(M)}(\pi \otimes \pi_0, \mathbb{C}) \neq 0$.

This determines a character $\chi_{\pi_0}$ of $A_{\varphi_0}$ and the association $\pi_0 \mapsto \chi_{\pi_0}$ gives a bijection between the $L$-packet $\Pi(\varphi_0)$ and the set of irreducible characters of $A_{\varphi_0}$.

In the conjecture of Langlands and Vogan as given in Section 4, the elements in an $L$-packet $\Pi(\varphi)$ are parametrized by characters of the component group $A_\varphi$. The above conjecture says that one can alternatively use the collection of epsilon factors described above to serve as parameters for elements of $\Pi(\varphi)$. In the next section, we will present a refined local conjecture by specifying the distinguished character $\chi$ above. This recipe uses symplectic root numbers for the representation $U = \Ind(V_0 \otimes V)$ of $W'_k$.

7. The refined local conjecture – the definition of $\chi$

We are now going to specify the character $\chi$ of $A_\varphi$ in Conjecture 6.1, up to a small ambiguity.

Assume that $\text{char}(k) \neq 2$ for simplicity, and write $K = k + ke$ with $\tau(e) = -e$. Then $e$ is the unique element of order 2 in the one dimensional torus $U(1) = K^\times/k^\times$ over $k$. Let $\omega_{K/k} : k^\times/NK^\times \xrightarrow{\sim} \pm 1$ be the quadratic character given by the class field theory.

Let $V$ be a representation of $W'(K)$ which satisfies

$$V^* \cong V'^{\vee}$$

as in § 3. Assume that we have a non-degenerate bilinear form $B : V \times V \rightarrow \mathbb{C}$ which satisfies

$$B(\sigma v, s \sigma s^{-1} w) = B(v, w)$$

$$B(w, s^2 v) = cB(v, w)$$

(7.1)
with \( c = c(V) = \pm 1 \). The induced representation \( \text{Ind}(V) \) of \( W'(k) \) is then \( c(V) \)-symmetric.

The 1-dimensional representations \( V \) satisfying (7.1) correspond to certain characters \( \alpha : K^\times \to \mathbb{C}^\times \), by local class field theory. Since \( \alpha^r = \alpha^{-1} \), the kernel of \( \alpha \) must contain the subgroup \( NK_k^\times \). Moreover, the restriction of \( \alpha \) to \( k^\times \) is given by

\[
\text{Res}(\alpha) = \begin{cases} 
1 & \text{if } c(\alpha) = 1 \\
\omega_{K/k} & \text{if } c(\alpha) = -1.
\end{cases}
\]

If \( V \) satisfies (7.1), so does the 1-dimensional representation \( \text{det}(V) \), and we have

\[
c(\text{det}(V)) = c(V)^{\dim(V)} \text{ in } \pm 1.
\]

In particular, if \( c(V)^{\dim(V)} = 1 \), \( \text{det}(V) \) is a character of \( k^\times/k^\times \), and may be evaluated on the involution \( e \).

We now assume that \( c(V) = -1 \), so \( \text{Ind}(V) \) is symplectic. In this case, we may define the local root number [Gr, Motives]

\[
\epsilon(\text{Ind}V) = \epsilon(\text{Ind}V, \psi, dx, \frac{1}{2})
\]

where \( \psi \) is any non-trivial additive character of \( k \) and \( dx \) is the unique Haar measure which is self-dual for Fourier transform with respect to \( \psi \). Then

\[
\epsilon(\text{Ind}V) = \pm 1,
\]

and

\[
\epsilon(\text{Ind}(V \oplus W)) = \epsilon(\text{Ind}V)\epsilon(\text{Ind}W).
\]

Fix the choice of an auxiliary character,

\[
\begin{cases}
\mu : K^\times \to \mathbb{C}^\times \\
\text{Res}(\mu) = \omega_{K/k}.
\end{cases}
\]

Then \( c(V \otimes \mu) = 1 \) and we may define the sign

\[
\det(V \otimes \mu)(e) = \pm 1.
\]

Then \( \det((V \oplus W) \otimes \mu)(e) = \det(V \otimes \mu)(e) \cdot \det(W \otimes \mu)(e) \).

If \( V \) has even dimension, the sign of \( \det(V \otimes \mu)(e) \) is independent of the choice of \( \mu \).

Indeed, in this case we find

\[
\det(V \otimes \mu)(e) = \det(V(e) \cdot \mu^{\dim V}(e)) = \det(V(e) \cdot \mu^{\dim V}(e^2)) = \det(V(e) \cdot \mu^{\dim V}(e^2)(-1) \text{ as } e^2 = -e\bar{e}) = \det(V(e) \cdot \omega_{K/k}(-1)^{\dim V} \text{ as } -1 \in k^\times.
\]

(7.2)

We now give our first (2 term) formula for \( \chi \). Let \( \varphi \) be a parameter for \( U(W) \times U(W_0) \). Then

\[
\varphi_K = \text{Res}_K \varphi : W'_K \to \text{GL}(V_1) \times \text{GL}(V_2)
\]
with $V_1$ even dimensional and $V_2$ odd dimensional. The representation $V = V_1 \otimes V_2$ satisfies (7.1) with $c(V) = -1$.

Let $a$ be an involution in the centralizer $C_\varphi$ of $\varphi$ in $\text{GL}(V_1) \times \text{GL}(V_2)$. Then $V$ decomposes as a direct sum of eigenspaces

$$V = V^{a=1} \oplus V^{a=-1}.$$ Both summands satisfy (7.1) with $c(V) = -1$, and we define

$$\chi(a) = \epsilon(\text{Ind}(V^{a=-1})) \cdot \det(V^{a=-1} \otimes \mu)(e).$$

(7.3)

In the next section, we are going to show that

$$\chi(ab) = \chi(a)\chi(b)$$

whenever $a$ and $b$ are commuting involutions in $C_\varphi$, and that $\chi(a)$ depends only on the image of $a$ in $C_\varphi/C_\varphi^0 = A_\varphi$. We assume this for the rest of this section. This gives the desired character (up to the choice of $\mu$)

$$\chi : A_\varphi \to \langle \pm 1 \rangle.$$ From (7.2), we obtain,

**Corollary 7.1.** If $W = V^{a=-1}$ has even dimension, then

$$\chi(a) = \epsilon(\text{Ind}W) \det W(e)\omega_{K/k}(-1)^{\frac{1}{2}\dim W}.$$ In this case, the value $\chi(a)$ is independent of the choice of the auxiliary character $\mu : K^\times/NK^\times \to \mathbb{C}^\times$.

In the situation of Corollary 7.1, we can find an equivalent formula for $\chi(a)$ which involves only local root numbers. Indeed, let $W$ be a representation of $W''(K)$ satisfying (7.1) with $c = -1$, and assume that $\dim W \equiv 0 \mod 2$. Then $\det W$ is a character of $K^\times/k^\times$ and $\text{Ind}(W)$ is symplectic.

Let $\psi_0$ be a non-trivial additive character of $K$ which is trivial on $k$. Then

$$\psi_0(x) = \psi \circ \text{Tr}(ex)$$

for a non-trivial additive character $\psi$ of $k$. Let $dx_0$ be the unique Haar measure on $K$ which is self-dual with respect to $\psi_0$, and define

$$\epsilon_0(W) = \epsilon(W, \psi_0, dx_0, \frac{1}{2}).$$

Using formal properties of local constants, as well as their inductivity in dimension 0, we can show

$$\epsilon_0(W) = \epsilon(\text{Ind}W) \det W(e)\omega_{K/k}(-1)^{\frac{1}{2}\dim W}.$$ (7.4)

Hence the formula in Corollary 7.1 becomes

$$\chi(a) = \epsilon_0(V^{a=-1}).$$
Before giving the proof that $\chi(a)$ depends only on the image of the involution $a$ in $A_{\varphi} = C_{\varphi}/C_{\varphi}^{0}$, we present some other formulae for $\chi(a)$ using the decomposition

$$a = a_1 \times a_2 \quad \text{in} \quad C_{\varphi} = C_1 \times C_2,$$

where $C_1 \subset \text{GL}(V_1)$ is the centralizer of $\varphi_K$. and $C_2 \subset \text{GL}(V_2)$ is the centralizer of $\varphi_K$. Since $V = V_1 \otimes V_2$ is a tensor product, we have

$$V^{\varphi=1} = (V_1^{\varphi_1=1} \otimes V_2^{\varphi_2=1}) \oplus (V_1^{\varphi_1=1} \otimes V_2^{\varphi_2=-1})\oplus (V_1^{\varphi_1=1} \otimes V_2^{\varphi_2=1}) \oplus (V_1^{\varphi_1=1} \otimes V_2^{\varphi_2=-1}) - 2(V_1^{\varphi_1=-1} \otimes V_2^{\varphi_2=-1}).$$

Since all of these representations have $\epsilon$ and $\det$ invariants in $\pm 1$, which are homomorphisms on the Grothendieck group, we find the (4 term) formula:

$$\chi(a_1 \times a_2) = \epsilon(\text{Ind}(V_1^{\varphi_1=1} \otimes V_2)) \cdot \det(V_1^{\varphi_1=1} \otimes V_2 \otimes \mu)(\epsilon)$$

$$\epsilon(\text{Ind}(V_1 \otimes V_2^{\varphi_2=-1})) \cdot \det(V_1 \otimes V_2^{\varphi_2=-1} \otimes \mu)(\epsilon).$$

We have the congruences:

$$\dim V_1 \equiv 0 \mod 2$$
$$\dim V_2 \equiv 1 \mod 2$$
$$\dim V_1^{\varphi_1=1} \equiv \dim V^{\varphi=1} \mod 2.$$

If $\dim V^{\varphi=1}$ is even, we have the following formula for (7.5), using the argument of (7.4)

$$\chi(a_1 \times a_2) = \epsilon_0(V_1^{\varphi_1=1} \otimes V_2)\epsilon_0(V_1 \otimes V_2^{\varphi_2=-1}).$$

If we expand the determinants in formula (7.5), we obtain a 6-term formula:

$$\chi(a_1 \times a_2) = \epsilon(\text{Ind}(V_1^{\varphi_1=1} \otimes V_2)) \cdot \det(V_1^{\varphi_1=1} \otimes V_2 \otimes \mu)(\epsilon)^{\dim V_2} \det V_2(\epsilon)^{\dim V_1^{\varphi_1=1}}$$
$$\epsilon(\text{Ind}(V_1 \otimes V_2^{\varphi_2=-1})) \cdot \det(V_1 \otimes \mu)(\epsilon)^{\dim V_2^{\varphi_2=-1}} \det V_2^{\varphi_2=-1}(\epsilon)^{\dim V_1}.$$

Since $\det V_2^{\varphi_2=-1}(\epsilon) = \pm 1$, we find the last term

$$\det V_2^{\varphi_2=-1}(\epsilon)^{\dim V_1} = 1.$$

Also the third term is trivial when $\det V_2(\epsilon) = 1$. When $\det V_2(\epsilon) = -1$, it is given by the character

$$\eta_1(a_1) = (-1)^{\dim V_1^{\varphi_1=1}}.$$
8. \( \chi \) is a character of \( A_\varphi \)

Let \( \varphi \) be a Langlands parameter for \( U(W) \times U(W_0) \), and let \( \varphi_K : W'_K \to \text{GL}(V_1) \times \text{GL}(V_2) \) be its restriction to the Weil-Deligne group of \( K \). We assume \( V_1 \) has even dimension and \( V_2 \) has odd dimension, and define the representation

\[
V = V_1 \otimes V_2.
\]

For an involution \( a = a_1 \times a_2 \) in the centralizer \( C_\varphi = C_1 \times C_2 \) of \( \varphi \) in \( \text{GL}(V_1) \times \text{GL}(V_2) \), we have defined the sign \( \chi \) in (7.3). This definition uses an auxiliary character \( \mu : K^\times \to \mathbb{C}^\times \) with \( \text{Res}(\mu) = \omega_{K/k} \) on \( k^\times \), but when \( V^{a=1} \) has even dimension, the sign \( \chi(a) \) is independent of the choice of \( \mu \).

In this section, we will show that \( \chi(a) \) depends only on the image of \( a \) in the quotient group \( A_\varphi = C_\varphi/C_\varphi^0 \), and that the resulting function on \( A_\varphi \) is a homomorphism

\[
\chi : A_\varphi \to (\pm 1).
\]

We will also describe the dependence of the character \( \chi \) on the choice of \( \mu \).

When the involutions \( a \) and \( b \) in \( C_\varphi \) commute, we have the identity

\[
\chi(ab) = \chi(a)\chi(b) \quad \text{in} \quad (\pm 1).
\]

This follows from the identity,

\[
V^{ab=1} = V^{a=1} + V^{b=1} - 2V^{a=1}V^{b=1}
\]

in the representation ring of \( W'_K \). Since commuting involutions generate the cosets of \( C_\varphi^0 \) in \( C_\varphi \), it suffices to show that \( \chi(a) \) depends only on the image of \( a \) in \( A_\varphi \).

Using the description of \( C_\varphi = C_1 \times C_2 \) in §3, we see that there are 6 cases to check:

1. \( a = a_1 \times 1 \) with \( a_1 \in \text{GL}(n) \).
2. \( a = a_1 \times 1 \) with \( a_1 \in \text{Sp}(2n) \).
3. \( a = a_1 \times 1 \) with \( a_1 \in \text{O}(n) \).
4. \( a = 1 \times a_2 \) with \( a_2 \in \text{GL}(n) \).
5. \( a = 1 \times a_2 \) with \( a_2 \in \text{Sp}(2n) \).
6. \( a = 1 \times a_2 \) with \( a_2 \in \text{O}(n) \).

In cases (1), (2), (4), (5) we must show that \( \chi(a) = 1 \). In cases (3) and (6) we need to show that \( \chi(a) \) is a function of \( \det(a_1) \) or \( \det(a_2) \) respectively.

To check this, we will use the formulae,

\[
\begin{align*}
\chi(a_1 \times 1) &= \epsilon(\text{Ind}(V_1^{a_1=1} \otimes V_2)) \cdot \det(V_1^{a_1=1} \otimes \mu)(e)^{\dim V_2} \det(V_2)(e)^{\dim V_1^{a_1=1}} \cdot \\
\chi(1 \times a_2) &= \epsilon(\text{Ind}(V_1 \otimes V_2^{a_2=1}) \cdot \det(V_1)(e)^{\dim V_2^{a_2=1}}.
\end{align*}
\]

In cases (1) and (2), we have \( \dim V_1^{a_1=1} \equiv 0 \mod 2 \), so

\[
\chi(a_1 \times 1) = \epsilon(\text{Ind}(V_1^{a_1=1} \otimes V_2)) \cdot \det(V_1^{a_1=1})(e)^{\omega_{K/k}(-1)^{\frac{1}{2}}}^{\dim V_1^{a_1=1}}.
\]
In cases (4) and (5), we have \( \dim V_{2}^{a_{2}=-1} \equiv 0 \mod 2 \), so

\[
\chi(1 \times a_{2}) = \epsilon(\text{Ind}(V_{1} \otimes V_{2}^{a_{2}=-1})).
\]

(1) Let \( 0 \leq p \leq n \) be the number of \(-1\)'s in \( a_{1} \). Then,

\[
V_{1}^{a_{1}=-1} = p(W + W')
\]

where \( W \) is the irreducible summand of \( V_{1} \) and \( W' = (W^{*})' \). Hence the \( \epsilon \)-factor is,

\[
\epsilon(\text{Ind}(W \otimes V_{2}))^{p} = \det(\text{Ind}(W \otimes V_{2}))(1)^{p} = (\det(W \otimes V_{2})(1) \cdot \omega_{K/k}(-1)^{\dim W \otimes V_{2}})^{p} = (\det W(-1) \cdot \omega_{K/k}(-1)^{\dim W})^{p}.
\]

The other terms in \( \chi(a) \) are

\[
\det W \cdot \det W'(e)^{p} = \det W(-1)^{p} \cdot \omega_{K/k}(-1)^{\dim W}.
\]

Hence \( \chi(a) = 1 \).

(2) Suppose \(-1\) appears with multiplicity \( 2p \), with \( 0 \leq p \leq n \) in \( a_{1} \), and

\[
V_{1}^{a_{1}=-1} = 2pW,
\]

with \( W \) an irreducible summand of \( V_{1} \) of type 2. The \( \epsilon \)-factor is

\[
\epsilon(\text{Ind}(W \otimes V_{2}))^{2p} = \det(\text{Ind}(W \otimes V_{2}))(1)^{p} = (\det(W \otimes V_{2})(1)^{p} \cdot \omega_{K/k}(-1)^{p \cdot \dim W \otimes V_{2}})^{p} = (\det W(-1)^{p} \cdot \omega_{K/k}(-1)^{p \cdot \dim W})^{p}.
\]

Again this cancels with other factors in \( \chi(a) \).

(3) Let \( p \) be the number of \(-1\)'s in \( a_{1} \), so \( a_{1} \in \text{SO}(n) \) if and only if \( p \) is even. Here,

\[
V_{1}^{a_{1}=-1} = p \cdot W
\]

where \( W \) is an irreducible summand of \( V_{1} \) of type 3. The \( \epsilon \)-factor is

\[
\epsilon(\text{Ind}(W \otimes V_{2}))^{p},
\]

which depends only on the parity of \( p \), and is 1 if \( p \) is even. The other factors are,

\[
\det(W \otimes \mu)(e)^{p},
\]

\[
\det(V_{2})(e)^{p \cdot \dim W}.
\]

Again these depend only on the parity of \( p \), and are 1 when \( p \) is even.

(4) The proofs in cases (4), (5), (6) are similar, and we leave them to the reader.
What is the dependence of the character $\chi$ of $A_\varphi$ on the choice of $\mu$? If 
\[ \mu' = \gamma \cdot \mu, \]
is another choice, with $\gamma : K^\times/k^\times \to \mathbb{C}^\times$, we find 
\[ \chi'(a) = \gamma(e)^{\dim V_{a^\varphi} = -1} \cdot \chi(a) \]
Hence $\chi'$ is either equal to $\chi$ in $\text{Hom}(A_\varphi, \pm 1)$, or to the character $\chi \cdot (\eta_1 \times 1)$, where 
\[ \eta_1 : A_1 : \to (\pm 1) \]
\[ a_1 \to (-1)^{\dim V_{a^\varphi} = -1} \]
is the generic character defined in §3.

9. Alternative Description of Recipe

The recipe for the character $\chi$ given in the refined local conjecture of the previous sections is along the lines of that given in [GP1] for the orthogonal groups. In this section, we will give an alternative description of the recipe which is perhaps more transparent. The recipe given in this section also depends on the choice of a character $\mu : K^\times \to \mathbb{C}^\times$ as in the last section.

We begin by giving this alternative description in the orthogonal case.

Orthogonal groups

We reinterpret the recipe in the paper [GP1] which, to a pair $(\Sigma_1, \Sigma_2)$ consisting of an orthogonal parameter 
\[ \Sigma_1 : W'_k \to O(2m, \mathbb{C}), \]
and a symplectic parameter 
\[ \Sigma_2 : W'_k \to \text{Sp}(2n, \mathbb{C}), \]
attaches a character 
\[ \epsilon_{\Sigma_1, \Sigma_2} : A_{\Sigma_1} \times A_{\Sigma_2} \to \{ \pm 1 \}. \]
Here, $A_{\Sigma_1}$ and $A_{\Sigma_2}$ are the component groups of $\Sigma_1$ and $\Sigma_2$ respectively. We recall that the component group $A_{\Sigma_1}$ (resp. $A_{\Sigma_2}$) is a free abelian group over $\mathbb{Z}/2$, with basis indexed by the distinct irreducible representations $\sigma_1$ (resp. $\sigma_2$) of $W'_k$ appearing in $\Sigma_1$ (resp. $\Sigma_2$) which are orthogonal (resp. symplectic).

We first define a tentative character $\epsilon_{\Sigma_1, \Sigma_2}^{\text{tent}}$ on the free abelian group over $\mathbb{Z}/2$ generated by such $\sigma_1$’s and $\sigma_2$’s. This is given by:

\[
\begin{align*}
\epsilon_{\Sigma_1, \Sigma_2}^{\text{tent}}(\sigma_1) &= \epsilon(\sigma_1 \otimes \Sigma_2, \psi); \\
\epsilon_{\Sigma_1, \Sigma_2}^{\text{tent}}(\sigma_2) &= \epsilon(\Sigma_1 \otimes \sigma_2, \psi).
\end{align*}
\]
We shall need to modify the tentative character $\epsilon_{\Sigma_1, \Sigma_2}^{\text{tent}}$, which is supposed to detect the non-vanishing of certain linear forms in the Gross-Prasad conjecture, so that it is manifestly trivial when it is supposed to be. For this we employ the following heuristic, which should be possible to prove, but we have not managed to do so:

1. An irreducible generic principal series representation of a split orthogonal group $\text{SO}(n+1)$ always contains the generic member of a generic $L$-packet of $\text{SO}(n)$ as a quotient.

2. The generic member of a generic $L$-packet of $\text{SO}(n+1)$ contains an irreducible principal series representation of the split $\text{SO}(n)$ as a quotient.

This heuristic suggests that we define, $\epsilon_{\Sigma_1, \Sigma_2}$ by the following modifications:

$$
\left\{
\begin{array}{ll}
\epsilon_{\Sigma_1, \Sigma_2}^{\text{tent}}(\sigma_1) = \epsilon(\sigma_1 \otimes \Sigma_2)\epsilon(\sigma_1 \otimes P\sigma_2) \\
\epsilon_{\Sigma_1, \Sigma_2}^{\text{tent}}(\sigma_2) = \epsilon(\Sigma_1 \otimes \sigma_2)\epsilon(\Sigma_1 \otimes ps_2)
\end{array}
\right.
$$

where $P\sigma_2$ is the parameter of a principal series representation on $\text{SO}(2n+1)$, and $ps_2$ is the parameter of a principal series representation on $\text{SO}(2d_2+1)$ where $\sigma_2$ has dimension $2d_2$.

With this modification, $\epsilon_{\Sigma_1, \Sigma_2}$ becomes the trivial character when it is expected to be. Moreover, we have:

**Lemma 9.1.** The character $\epsilon_{\Sigma_1, \Sigma_2}$ defined above agrees with the character defined in [GP1].

**Remark:** For comparing the character defined here with that in [GP1], note that for a principal series representation of $\text{SO}(2n+1)$ with parameter $ps = \tau \oplus \tau^\vee$ with $\det(\tau(-1)) = 1$,

$$
\epsilon(\sigma \otimes ps) = \det \sigma (-1)^n.
$$

**Unitary groups**

Now we come to the unitary case, which is the subject matter of this paper. Let $\sigma$ be an irreducible representation of $W'_K$:

$$
\sigma : W'_K \to \text{GL}_m(\mathbb{C})
$$

where $K$ is a quadratic extension of a local field $k$ such that

$$
\bar{\sigma} \cong \sigma^\vee
$$

where $\bar{\sigma}$ denotes the representation of $W'_K$ obtained by conjugating $\sigma$ by an element of $W'_k \setminus W'_K$. Then one knows that one of the following holds:
(1) $m$ is even, in which case $\det \sigma$ is trivial on $k^\times$, and $\text{Ind}_{W_k^\prime}^{W_k} \sigma$ is a symplectic representation of $W_k^\prime$;
(2) $m$ is odd, in which case $\det \sigma$ is trivial restricted to $k^\times$ if and only if $c(\sigma) = 1$; if $\det \sigma$ is nontrivial, it is $\omega_{K/k}$.

We next turn to the local root numbers, where we will only be interested in representations $\sigma$ with $\bar{\sigma} \cong \sigma^\vee$. Let $\psi$ be a non-trivial character on $K$ which is trivial on $k$.

**Lemma 9.2.** For a representation $\sigma$ of $W_K^\prime$ with $\bar{\sigma} \cong \sigma^\vee$, character $\chi_\sigma$ of $K^\times$ such that $\chi_\sigma = \det \sigma$ on $k^\times$, and $\psi$ a nontrivial character on $K$ which is trivial on $k$,

$$\epsilon(\sigma, \psi) \cdot \epsilon(\chi_\sigma, \psi) = \pm 1,$$

and is independent of $\psi$.

**Proof:** This follows from the fact that such characters $\psi$ form a principal homogeneous space over $k^\times$, and the following generalities:

(1) $\epsilon(\sigma, \psi_a) = (\det \sigma)(a) \cdot \epsilon(\sigma, \psi)$.
(2) $\epsilon(\sigma, \psi) = \epsilon(\bar{\sigma}, \bar{\psi})$.
(3) $\epsilon(\sigma, \psi) \cdot \epsilon(\sigma^\vee, \psi_{-1}) = 1$.

**Remark:** By a theorem due to Fröhlich and Queyrut, for a character $\psi$ of $K$ which is trivial on $k$, $\epsilon(\chi_\sigma, \psi) = 1$ if $\chi_\sigma$ is trivial on $k^\times$, and therefore this can be taken to be a ‘correction’ term introduced to make $\epsilon(\sigma, \psi)$ independent of $\psi$.

Let $V_1$ be an even dimensional Hermitian space over $K$ and $V_2$ an odd dimensional Hermitian space over $K$. Let $U(V_1)$ and $U(V_2)$ be the corresponding unitary groups over $k$, and let $\varphi_1$ and $\varphi_2$ be generic Langlands parameters for $U(V_1)$ and $U(V_2)$. Write $\Sigma_1$ and $\Sigma_2$ for the restriction of $\varphi_1$ and $\varphi_2$ to $W_K^\prime$. As described in Section 3, the component group $A_{\varphi_1}$ (resp. $A_{\varphi_2}$) is an elementary abelian 2-group, with basis given by the set of irreducible conjugate-self-dual representations $\sigma_1$ (resp. $\sigma_2$) of $W_K^\prime$ appearing in $\Sigma_1$ (resp. $\Sigma_2$) which have the further property that $c(\sigma_1) = -1$ (resp. $c(\sigma_2) = 1$).

Let’s note that

$$\det(\Sigma_1 \otimes \sigma_2) = \det(\Sigma_1)^{\dim \sigma_2} \cdot \det(\sigma_2)^{\dim \Sigma_1} = \nu$$

with $\nu$ a character of $K^\times$ with trivial restriction to $k^\times$ as $\Sigma_1$ is even dimensional of trivial determinant restricted to $k^\times$. Also,

$$\det(\sigma_1 \otimes \Sigma_2) = \det(\sigma_1)^{\dim \Sigma_2} \cdot \det(\Sigma_2)^{\dim \sigma_1} = \nu'$$

with $\nu'$ a character of $K^\times$ whose restriction to $k^\times$ is the same as the restriction of $\det \sigma_1$ to $k^\times$ as $\Sigma_2$ is odd dimensional of trivial determinant restricted to $k^\times$.

We define a character $\epsilon_{\varphi_1, \varphi_2}^{\text{tent}}$ on $A_{\varphi_1} \times A_{\varphi_2}$, which has basis given by the $\sigma_1$’s and $\sigma_2$’s as above, by
where $\chi_{\sigma_1}$ is the trivial character of $K^\times$ if $\det \sigma_1$ is trivial restricted to $k^\times$, and is $\mu$ if $\det \sigma_1$ is non-trivial restricted to $k^\times$. (We recall having fixed a character $\mu: K^\times \to \mathbb{C}^\times$ in the last sections whose restriction to $k^\times$ is $\omega_{K/k}$.)

By the lemma above, $\epsilon_{\varphi_1,\varphi_2}^{\text{tent}}$ is independent of the choice of the additive character $\psi$ of $K$ trivial on $k$. Now, as in the orthogonal case, let us see if we need to modify our recipe based on the following suggestions about principal series representations of unitary groups.

(1) A principal series representation of the quasi-split group $U(2n)$ contains the $f$-generic member of a generic $L$-packet on $U(2n-1)$.

(2) An irreducible principal series representation of the quasi-split group $U(2n)$ is contained in the $f$-generic member of an $L$-packet on $U(2n+1)$.

Remark: It is worth noting that these statements about principal series representations of $U(n)$ are sensitive to the parity of $n$. For example it is not true that principal series representation of $U(3)$ contains the generic member of an $L$-packet on $U(2)$, for example, because the $U(2)$ considered might be compact!

We now calculate our character $\epsilon_{\varphi_1,\varphi_2}^{\text{tent}}$ for the case when $\varphi_1$ corresponds to an irreducible principal series representation for $U(2d)$. Recall that the maximal torus of the Borel subgroup $B$ in $U(2d)$ can be taken to be $K^\times \times \cdots \times K^\times$, and thus a principal series representation on $U(2d)$ is parametrized by a $d$-tuple of characters $(\chi_1, \cdots, \chi_d)$ of $K^\times$. The parameter $\varphi_1$ of the corresponding principal series representation of $U(2d)$ has restriction to $W'_K$ given by:

$$\Sigma_1 = \chi_1 \oplus \cdots \oplus \chi_d \oplus \chi_1^{-1} \oplus \cdots \oplus \chi_d^{-1} = \tau \oplus \tau^\vee.$$  

We now calculate the local root numbers using the following lemma.

**Lemma 9.3.** For a representation $\varphi$ of $W'_K$,

$$\epsilon(\varphi, \psi) \cdot \epsilon(\varphi^\vee, \psi) = 1.$$

**Proof:** This follows from generality about epsilon factors:

$$\epsilon(\varphi, \psi) \cdot \epsilon(\varphi^\vee, \psi) = \epsilon(\varphi, \psi) \cdot \epsilon(\varphi^\vee, \tilde{\psi})$$

$$= \epsilon(\varphi, \psi) \cdot \epsilon(\varphi^\vee, \psi_1)$$

$$= \det \varphi(-1) \cdot \epsilon(\varphi, \psi) \cdot \epsilon(\varphi^\vee, \psi)$$

$$= 1.$$
Note now that the determinant of a representation of the form $\pi \otimes (\tau \oplus \bar{\tau})$ with $\pi \cong \bar{\pi}$ is of the form $\chi / \bar{\chi}$, and by the theorem of Fröhlich-Queyrut, $\epsilon(\chi / \bar{\chi}, \psi) = 1$. Therefore for $\varphi_1$ as above, we have:

$$\epsilon_{\varphi_1,\varphi_2}^{\text{tent}} = 1.$$ 

This means that unlike the orthogonal case, we do not need any modification to $\epsilon_{\varphi_1,\varphi_2}^{\text{tent}}$, and we simply set

$$\epsilon_{\varphi_1,\varphi_2} = \epsilon_{\varphi_1,\varphi_2}^{\text{tent}}.$$ 

The character $\epsilon_{\varphi_1,\varphi_2}$ is the same as the one described in the refined local conjecture except for the possible ambiguity between $\chi$ and $\chi \cdot (\eta_1 \times 1)$.

10. Unramified parameters

In this section, we assume that $k$ is non-Archimedean. We determine the component groups $A_\varphi$ and the character $\chi$ of $A_\varphi$ for parameters $\varphi$ whose restriction to $W'_K$

$$\varphi_K : W'_K \to \text{GL}(V_0) \times \text{GL}(V_1)$$

is unramified. We call these unramified parameters –although the corresponding $L$-packets $\pi$ contain unramified representations only when the quadratic extension $K/k$ is also unramified.

First consider a single unitary group $U(W)$, where the split Hermitian space $W$ has dimension $n$ over $K$. Then an unramified parameter is completely determined by the conjugacy class of the semi-simple element

$$\varphi_K(Fr) = \begin{pmatrix} \alpha_1 & \ldots & \ldots & \ldots \\ \ldots & \alpha_i & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \alpha_n \end{pmatrix}$$

in $\text{GL}(V) = \text{GL}_n(\mathbb{C})$. The irreducible submodules $V_i$ all have dimension 1; since $\text{ord}(x) = \text{ord}(sx)$ for $x \in K^\times$, we have $V_i^x \cong V_i$ for all $i$. Hence an analysis of the types of submodules, as in section 3, shows that either $\alpha_i^2 = 1$, or there is an eigenvalue $\alpha_j$ with $\alpha_i \cdot \alpha_j = 1$. Moreover, the eigenvalues $\alpha_i$ and $\alpha_j$ occur with the same multiplicity in $V$.

The component group $A_\varphi$ is determined by the multiplicities of the eigenvalues $\alpha_i = \pm 1$.

Assume first that $K/k$ is unramified. Then the eigenvalue $-1$ corresponds to the unramified quadratic character $\mu$ of $K^\times$, which satisfies $\text{Ver}(\mu) = \omega_{K/k}$. The eigenvalue 1 corresponds to the trivial character of $K^\times$, which has $\text{Ver}(1) = 1$. These are the only unramified characters of $K^\times$ which are trivial on $NK^\times$. Since $\text{det } V$ is an unramified character with $\text{Ver}(\text{det } V) = 1$, we must therefore have $\text{det } V = 1$. 

Unramified parameters
Proposition 10.1. When \( K/k \) is unramified, the eigenvalue \(-1\) occurs with even multiplicity in \( V \), and the eigenvalue 1 occurs with multiplicity \( \equiv n \mod 2 \) in \( V \). We have:

1. When \( n \) is odd, the component group \( A_\varphi = \mathbb{Z}/2 \), and is generated by the image of \(-1\) in \( Z(LG) \).
2. When \( n \) is even, the component group is given by

\[
A_\varphi = \begin{cases} 
1 & \text{if the multiplicity of } -1 \text{ is zero in } V \\
\mathbb{Z}/2 & \text{if } -1 \text{ occurs as an eigenvalue in } V, 
\end{cases}
\]

and the image of \(-1\) in \( Z(LG) \) is trivial in \( A_\varphi \).

Next assume that \( K/k \) is ramified. Then the only unramified characters of \( K^\times \) which are trivial on \( NK^\times \) are again the trivial and quadratic character. In this case, both are trivial on \( k^\times \), so there are no unramified characters \( \mu \) with \( \text{Ver}(\mu) = \omega_{K/k} \).

Proposition 10.2. When \( K/k \) is ramified and \( n \) is even, the eigenvalues \( \pm 1 \) both appear with even multiplicity in \( V \). Hence \( \det V = 1 \), and \( A_\varphi = 1 \).

If \( n \) is odd, one of the eigenvalues \( \pm 1 \) appears with even multiplicity and the other appears with odd multiplicity in \( V \), depending on the character \( \det V \), with \( (\det V)^2 = 1 \). We have

\[
A_\varphi = \begin{cases} 
\mathbb{Z}/2 & \text{if one of these eigenvalues appears in } V \\
(\mathbb{Z}/2)^2 & \text{if both do} 
\end{cases}
\]

We now consider generic unramified parameters for the group \( G = U(W) \times U(W_0) \), and investigate the character \( \chi \) of \( A_\varphi = A_1 \times A_2 \). Recall that \( \chi \) depends, up to multiplication by \( \eta_1 \times 1 \), on the choice of an auxiliary homomorphism \( \mu : K^\times \to \mathbb{C}^\times \) with \( \text{Ver}(\mu) = \omega_{K/k} \). When \( A_1 = 1 \), the character \( \chi \) is well-defined. This is the case when \( K/k \) is ramified. In the unramified case, we have

Proposition 10.3. Assume that \( K/k \) is unramified and that \( \varphi \) is an unramified generic parameter for \( G \). If we choose \( \mu \) to be the unique unramified character of \( K^\times \) with \( \text{Ver}(\mu) = \omega_{K/k} \), then \( \chi = 1 \).

Proof: We have \( A_1 \) of order 1 or 2, and \( A_2 \) is generated by the image of \(-1\) in the center. Since \(-1\) has trivial image in \( A_1 \), and \( \chi(-1,1) = \chi(1,-1) \), we see that \( \chi \) is trivial on the subgroup \( 1 \times A_2 \). It suffices to show it is trivial on the subgroup \( A_1 \times 1 \). If \( A_1 \) is non-trivial, a basis element \( a \) corresponds to the summand of \( V_1 \) which is the \(-1\) eigenspace. Then

\[
V_1^{a=-1} = \mu, \quad \det V_2 = 1,
\]

and

\[
\chi(a,1) = \epsilon(\text{Ind}((\mu \otimes V_2))\mu^2(e)).
\]

Since the epsilon factor of an unramified representation is 1, and \( \mu^2 = 1 \), we have \( \chi = 1 \).
The determination of $\chi$ (for a fixed choice of $\mu$) in Proposition 10.3 does not specify the representation $\pi$ in the $L$-packet with $\text{Hom}_H(\pi, \mathbb{C}) \neq 0$, as we have not chosen a generic character. Instead, we will specify $\pi$ using spherical vectors. There are two conjugacy classes of hyperspecial maximal compact subgroup in $G(k) = U(W_0) \times U(W)$, but there is a unique $H$-conjugacy class of such $J \subset G(k)$ with the additional property that $H(k) \cap J$ is hyperspecial in $H(k)$. If $W_0 = \langle e_n \rangle$ and we have chosen an orthogonal basis $\langle e_1, \cdots, e_{n-1} \rangle$ for $W_0$ with $\langle e_i, e_i \rangle = \langle e_n, e_n \rangle$ for all $i$, then $J$ is the subgroup of $U(W_0) \times U(W)$ which stabilizes the lattices $L_0 = \bigoplus_{i=0}^{n-1} A_k e_i$ and $L = \bigoplus_{i=0}^{n} A_k e_i$ in $W_0$ and $W$ respectively. See §18 for the discussion of all of the maximal compact subgroups in $G(k)$.

**Conjecture 10.4.** The unique representation $\pi$ of $G$ in the $L$-packet of a generic, unramified parameter with $\text{Hom}_H(\pi, \mathbb{C}) \neq 0$ is $J$-spherical: $\pi_J \neq 0$. Moreover, the pairing $\text{Hom}_J(\mathbb{C}, \pi) \times \text{Hom}_H(\pi, \mathbb{C}) \to \mathbb{C}$ is non-degenerate.

### 11. Discrete series parameters

We now consider certain discrete series $L$-packets for $G = U(W) \times U(W_0)$. In this case, all the irreducible factors $V_i$ in the Galois representations $V_1$ and $V_2$ are of type 3, and all have multiplicity $m_i = 1$. We will assume further that each irreducible factor has dimension 1, so

$$V_1 = \bigoplus V(\alpha_i)$$
$$V_2 = \bigoplus V(\beta_i)$$

where each $\alpha_i$ is a character of $K^\times / NK^\times$ with $\text{Ver}(\alpha_i) = \omega_{K/k}$, and each $\beta_i$ is a character of $K^\times / k^\times$. Since the multiplicities are 1, all of these characters are distinct, and we have the component groups

$$A_1 = (\mathbb{Z}/2)^{\dim V_1}$$
$$A_2 = (\mathbb{Z}/2)^{\dim V_2}$$

as large as possible.

These are the parameters of the discrete series representations for $k = \mathbb{R}$, and include the tamely ramified elliptic parameters when $k$ is non-Archimedean, $K/k$ is unramified, and Frobenius maps to $-1$ in the extended Weyl group $(S_n \times S_{n-1}) \cdot \text{Gal}(K/k)$. The group $A_\varphi$ has basis

$$\langle e_1, \cdots, e_{\dim V_1}; f_1, \cdots, f_{\dim V_2} \rangle$$

with

$$V_1^{e_i=-1} = V(\alpha_i)$$
$$V_2^{f_i=-1} = V(\beta_i).$$

Choosing an auxiliary character $\mu : K^\times \to \mathbb{C}^\times$ with $\text{Ver}(\mu) = \omega_{K/k}$, we find the formulae:
\[ \chi(e_i) = \prod_k \epsilon(\text{Ind} \alpha_i \beta_k) \cdot \alpha_i \mu(e) \cdot \prod_k \beta_k(e) \]
\[ \chi(f_i) = \prod_k \epsilon(\text{Ind} \alpha_k \beta_i) \cdot \prod_k \alpha_k(e) \cdot \epsilon_k(-1)^{\frac{1}{2} \dim V_i}. \]

From this it follows that
\[ \chi(e_i) \chi(e_j) = \prod_k \epsilon(\text{Ind} \alpha_i \beta_k) \epsilon(\text{Ind} \alpha_j \beta_k) \cdot \omega_{K/k}(e) \cdot \alpha_i \alpha_j(e) \]
\[ \chi(f_i) \chi(f_j) = \prod_k \epsilon(\text{Ind} \alpha_k \beta_i) \epsilon(\text{Ind} \alpha_k \beta_j) \]

independent of the choice of \( \mu \).

We now need a formula for the symplectic root number of the two dimensional representations
\[ \text{Ind}(\alpha \beta) \]
where \( \alpha \) and \( \beta \) are characters of \( K^\times \) with \( \text{Ver}(\alpha) = \omega_{K/k}, \text{Ver}(\beta) = 1 \). We achieve this in the following proposition when \( k \) is non-Archimedean.

**Proposition 11.1.** Let \( k \) be non-Archimedean, \( K/k \) is unramified and \( \alpha \) and \( \beta \) tamely ramified characters of \( K^\times \). Then \( \alpha \cdot \beta = \gamma \cdot \mu \), where \( \gamma \) is a tamely ramified character of \( K^\times/k^\times \) and \( \mu \) is the unramified quadratic character of \( K^\times \) (with \( \text{Ver}(\mu) = \omega_{K/k} \)).

Then:
1. If \( \gamma = 1 \), \( \epsilon(\text{Ind} \beta) = 1 \).
2. If \( \gamma \neq 1 \), \( \epsilon(\text{Ind} \beta) = -\gamma(e) \).

**Corollary 11.2.** Order the characters \( \alpha_i \) and \( \beta_i \) in the parameter \( \varphi \) so that
\[ \alpha_1 \cdot \beta_1 = \alpha_2 \cdot \beta_2 = \cdots, \alpha_p \cdot \beta_p = \mu, \]
and for no other products, \( \alpha_i \beta_j = \mu \). Then
\[ \begin{align*}
\chi(e_i) \chi(e_j) = 1 \quad &\text{if } i, j \leq p \text{ or } i, j > p \\
\chi(f_i) \chi(f_j) = -1 \quad &\text{otherwise.}
\end{align*} \]

Moreover, \( \chi(-1, 1) = \chi(1, -1) = (-1)^p \).

**Proof.** We have \( \epsilon(\text{Ind} \gamma) = \gamma(e) \) by the formula of Fröhlich-Queyrut [F-Q]. Since \( \mu \) is unramified, we find that
\[ \epsilon(\text{Ind} (\gamma \mu)) = \mu(f_\gamma) \cdot \gamma(e) \]
with \( f_\gamma \) the conductor of \( \gamma \), which is zero when \( \gamma = 1 \), and one when \( \gamma \neq 1 \), proving the proposition. \( \square \)
To prove the corollary, write
\[ \alpha_i \beta_j = \alpha'_i \beta_j \cdot \mu \]
where \( \alpha'_i \) is the restriction to the units, viewed as a character of \( U_K/U_k = K^\times/k^\times \). We have ordered our characters so that
\[ \gamma_{ij} = \alpha'_i \beta_j = 1 \]
precisely when \( i = j \leq p \).
For \( i, j \leq p \) we find that
\[ \chi(e_i) \chi(e_j) = \prod_{k \neq i} -\alpha'_i \beta_k(e) \prod_{k \neq j} -\alpha'_j \beta_k(e) \cdot \alpha_i \alpha_j(e) \]
as the products have an even number of terms.
For \( i, j > p \), we find,
\[ \chi(e_i) \chi(e_j) = \prod_k -\alpha'_i \beta_k(e) \prod_k -\alpha'_j \beta_k(e) \cdot \alpha_i \alpha_j(e) = 1 \]
as the products have an odd number of terms.
For \( i \leq p \), and \( j > p \), we find,
\[ \chi(e_i) \chi(e_j) = \prod_{k \neq i} -\alpha'_i \beta_k(e) \prod_k -\alpha'_j \beta_k(e) \cdot \alpha_i \alpha_j(e) \]
\[ = -\alpha'_j(e) \beta_i(e) \cdot \alpha_i \alpha_j(e) = -1. \]
Similar results hold good for \( i > p \) and \( j \leq p \), as well as for \( \chi(f_i) \chi(f_j) \), completing the proof of the corollary.

Now assume \( k = \mathbb{R} \), and \( K = \mathbb{C} \). Then \( e = i \) satisfies \( e^2 = -1 \), and the characters \( \alpha \) and \( \beta \) are given by the formulae
\[ \alpha(z) = (z/\bar{z})^\alpha = z^{2\alpha}/(\bar{z})^{\alpha} \quad \alpha \in \frac{1}{2} \mathbb{Z} - \mathbb{Z} \]
\[ \beta(z) = (z/\bar{z})^\beta \quad \beta \in \mathbb{Z}. \]
In particular, \( \beta(e) = (-1)^\beta \).

**Proposition 11.3.** We have the formulae:
\[ \epsilon(\text{Ind}(\alpha \beta)) = \begin{cases} (-1)^{\alpha+\beta+\frac{1}{2}} & \text{if } \alpha + \beta > 0 \\ -(-1)^{\alpha+\beta+\frac{1}{2}} & \text{if } \alpha + \beta < 0 \end{cases} \]
To calculate $\chi(e_i)\chi(e_j)$ and $\chi(f_i)\chi(f_j)$, we order the characters $\alpha_i$ and $\beta_i$ in the parameter $\varphi$ so that

$$
\alpha_1 > \alpha_2 > \alpha_3 \cdots \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}
$$

$$
\beta_1 > \beta_2 > \beta_3 \cdots \in \mathbb{Z}.
$$

**Corollary 11.4.** For $i < j$, we have

$$
\chi(e_i)\chi(e_j) = (-1)^{\#\{k: \alpha_i + \beta_k > 0 > \alpha_j + \beta_k\}}
$$

$$
\chi(f_i)\chi(f_j) = (-1)^{\#\{k: \beta_i + \alpha_k > 0 > \beta_j + \alpha_k\}}.
$$

**Proof.** The formula in the Proposition follows from Tate [Ta]. To obtain the formula in the corollary, we recall that

$$
\chi(e_i)\chi(e_j) = \prod_k \epsilon(\text{Ind}_{\alpha_i\beta_k})\epsilon(\text{Ind}_{\alpha_j\beta_k}) \cdot \mu^2\alpha_i\alpha_j(e).
$$

If $\alpha_i + \beta_k$ and $\alpha_j + \beta_k$ have the same sign, the $k^{th}$ term in the product is

$$
(-1)^{\alpha_i + \beta_k + \alpha_j + \beta_k + 1} = (-1)^{\alpha_i + \alpha_j}.
$$

Otherwise, it is the negative of this. Since there are an odd number of terms $k$, the product is equal to

$$
-(-1)^{\alpha_i + \alpha_j} \cdot (-1)^{\#\{k: \alpha_i + \beta_k > 0 > \alpha_j + \beta_k\}}.
$$

Since $\mu^2\alpha_i\alpha_j(e) = -(-1)^{\alpha_i + \alpha_j}$, this gives the first result.

Similarly,

$$
\chi(f_i)\chi(f_j) = \prod_k \epsilon(\text{Ind}_{\alpha_k\beta_i})\epsilon(\text{Ind}_{\alpha_k\beta_j})
$$

If $\beta_i + \alpha_k$ and $\beta_j + \alpha_k$ have the same sign, the $k^{th}$ term in the product is

$$
(-1)^{2\alpha_k + \beta_i + \beta_j + 1}.
$$

Otherwise, it is the negative of this. Since there are an even number of terms $k$ in the product, we find

$$
\chi(f_i)\chi(f_j) = (-1)^{\sum 2\alpha_k} \cdot (-1)^{\#\{k: \beta_i + \alpha_k > 0 > \beta_j + \alpha_k\}}.
$$

But $(-1)^{\sum 2\alpha_k} = \det V(-1)^2$. Since $\det V$ has trivial transfer, $\sum \alpha_k$ is an integer and $\det V(-1)^2 = 1$, completing the proof of the corollary.

Since we know how to describe the representations in the $L$-packets of discrete series parameters, the calculation of $\chi(e_i)\chi(e_j)$ and $\chi(f_i)\chi(f_j)$ allows us to say something about the representation $\pi = \pi(\varphi, \chi)$ of $G(k)$ with the (conjectural) $H(k)$-invariant linear form. We write $\pi = \pi_1 \otimes \pi_2$ following our notation $V = V_1 \otimes V_2$. In the case when $k = \mathbb{R}$, $\pi_1$ and $\pi_2$ are discrete series representations of even and odd dimensional unitary groups, with infinitesimal characters

$$
\alpha_1 > \alpha_2 > \cdots
$$

$$
\beta_1 > \beta_2 > \cdots
$$
in $X^* + \rho$ respectively. Moreover, in the chambers defined by their Harish-Chandra parameters, the simple root walls corresponding to

\[ e_i - e_{i+1} \text{ is compact } \iff \chi(e_i)\chi(e_{i+1}) = -1 \]
\[ f_i - f_{i+1} \text{ is compact } \iff \chi(f_i)\chi(f_{i+1}) = -1. \]

More generally, the root

\[ e_i - e_j \text{ is compact } \iff \chi(e_i)\chi(e_j) = (-1)^{i+j} \]
\[ f_i - f_j \text{ is compact } \iff \chi(f_i)\chi(f_j) = (-1)^{i+j}. \]

This determines $G(k)$, and in almost all cases $\pi$.

For example, if

\[ \alpha_1 > -\beta_d > \alpha_2 > -\beta_{d-1} > \cdots, \]

we find that $G(\mathbb{R}) = U(n) \times U(n-1)$ is compact and $\pi$ is finite dimensional, determined by its infinitesimal character. In this case, $\chi(e_i)\chi(e_j) = \chi(f_i)\chi(f_j) = (-1)^{i+j}$ for all $i, j$.

In the $p$-adic case, the representation $\pi$ in the $L$-packet of $\varphi$ all have the form

\[ \pi = \text{Ind}_M^G(R) \]

where $M$ is a compact parahoric subgroup containing the unramified elliptic torus $T = U(1)^{2n-1} \subset U(W') \times U(W'_0)$, and $R$ is a Deligne-Lusztig representation of the reductive quotient $M(q)$ of $M$. The cuspidal representation $R = R(T, \varphi)$ of $M(q)$ corresponds to the tame characters $\alpha$ and $\beta$ in the parameter $\varphi$, which give characters of $T(q)$ by local class field theory.

Our calculation of $\chi$ determines the root system of $M(q)$ over $\mathbb{F}_q^2$: $e_i - e_j$ and $f_i - f_j$ appear as roots if $\chi(e_i)\chi(e_j) = \chi(f_i)\chi(f_j) = 1$. From the corollary, we deduce that the derived group of $M(q)$ is isomorphic to

\[ (\text{SU}(p) \times \text{SU}(n-p)) \times (\text{SU}(p) \times \text{SU}(n-1-p)). \]

Moreover, the representation $R$ of $M(q)$ has the form

\[ (R_1 \otimes R_2) \otimes (R_1 \otimes R'_2), \]

where the characters $\alpha_i$ and $\beta_i$ giving the representations $R_2$ and $R'_2$ of $U_{n-p}(q)$ and $U_{n-p-1}(q)$ are all distinct.

12. The Global Conjecture

We now assume that $k$ is a global field, with char($k$) $\neq 2$. Let $\mathbb{A}$ denote the ring of adèles of $k$. Let $K$ be a quadratic field extension of $k$. We write $K = k + ke$ with $\text{Tr}(e) = 0$, and fix an auxiliary Hecke character $\mu : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ with

\[ \text{Ver}(\mu) = \omega_{K/k} : \mathbb{A}^\times / (\mathbb{NA}_K^\times \cdot k^\times) \to \{ \pm 1 \}. \]

Let $W_0 \subset W$ be a pair of split Hermitian spaces of dimensions $n - 1$ and $n$. Let $G = U(W) \times U(W_0)$ be the corresponding quasi-split group over $k$, with diagonally embedded
subgroup $H = U(W_0)$. We fix a generic homomorphism $f : N_G \to \mathbb{G}_a$, and compose with a character of $\mathbb{A}/k$ to get a generic character $\psi : N_G(\mathbb{A}) \to \mathbb{C}^\times$. Finally, let $\pi$ be an automorphic, generic representation of $G(\mathbb{A})$ which is tempered and appears in the discrete spectrum of $G$.

We write $\pi = \hat{\otimes} \pi_v$ as a restricted tensor product, where each $\pi_v$ is a $\psi_v$-generic irreducible representation of the local group $G(k_v)$. Let $\varphi_v$ be its Langlands parameter, and let $\pi'_v$ be the unique representation in the generic $L$-packet of $\varphi_v$ with

$$\text{Hom}_{H_v}(\pi'_v, \mathbb{C}) \neq 0.$$  

If $v$ splits in $K$, then $G(k_v) = \text{GL}_n(k_v) \times \text{GL}_{n-1}(k_v)$, and $\pi'_v = \pi_v$. If $v$ is not split in $K$, so $K_v$ is a field, we define the local character

$$\chi_v : A_{\varphi_v} \to \langle \pm 1 \rangle$$

using the auxiliary character $\mu_v$ and Proposition 7.1. Our local conjecture predicts that $\pi'_v$ corresponds to either $\chi_v$ or its product with $(\eta_1 \times 1)$.

In any case, $\pi'_v$ is an irreducible representation of the group $G'_v = U(M'_v) \times U(M'_{0,v})$, where $M'_{0,v} \subset M'_v$ is a pair of Hermitian spaces over $K_v$ with $M'_v/M'_{0,v} \cong (M/M_0) \otimes_K k_v$. A necessary condition for $M'_v \cong M \otimes_K k_v$, is that

$$\chi_v(-1, 1) = \chi_v(1, -1) = 1.$$  

When $v$ is non-Archimedean, this is also sufficient.

The representation $\pi_v$ and the local character $\mu_v$ are both unramified for almost all places $v$. At these places, $\pi'_v = \pi_v$ of $G'_v = G(k_v)$, and $\chi_v = 1$. This allows us to define the tensor product representation

$$\pi' = \hat{\otimes} \pi'_v$$

of the locally compact group

$$G'_k = \prod_v G'_v.$$  

The representation $\pi'$ is nearly equivalent to the generic representation $\pi$. It is the unique representation in this near equivalence class with

$$\text{Hom}_{H'_(\mathbb{A})}(\pi', \mathbb{C}) \neq 0.$$  

There are now three basic questions to address, each depending on the previous answer.

1. Is $G'_k$ the adelic points of a group $G' = U(M') \times U(M'_{0})$ defined over $k$, associated to a pair of Hermitian spaces $M'_{0} \subset M'$ with $M'/M'_{0} \cong M/M_0$?

2. If the answer to 1. is yes, so $G'_k = G'({\mathbb{A}})$ contains the discrete subgroup $G'(k)$, does the tempered representation $\pi'$ of $G'({\mathbb{A}})$ appear with multiplicity one in the discrete spectrum of $G'$?
3. If the answer to 2. is yes, so \( \pi' \) embeds uniquely up to scaling in a space of cuspidal functions \( f \) on \( G'(k) \backslash G'(\mathbb{A}) \), is the \( H'(\mathbb{A}) \)-linear form on \( \pi' \):

\[
f \rightarrow \int_{H'(k) \backslash H'(\mathbb{A})} f(h) \, dh
\]

non-zero in the one dimensional vector space \( \text{Hom}_{H'(\mathbb{A})}(\pi', \mathbb{C}) \neq 0 \)?

We will provide conjectural answers to these questions in order, using

1. the global root number \( \epsilon(\pi, V) \), associated to \( \pi \) and the symplectic representation \( V = \text{Ind}(V_1 \otimes V_2) \) of the \( L \)-group of \( G \) of dimension \( 2n(n-1) \).

2. the collection of global root numbers \( \epsilon(\pi, V^{a=-1}) \), for elements \( a \) in the centralizer of the global Langlands parameter.

3. the central critical \( L \)-value \( L(\pi, V, \frac{1}{2}) \).

To answer 1., let \( d(M'_v) \) be the Hermitian discriminant of the space \( M'_v \in k^x/NK^x_v \). In the non-Archimedean case, this invariant determines \( M'_v \) up to isomorphism. We have the formula:

\[
(d(M'_v), D)_v = \chi_v(-1, -1)(d(M), D)_v
\]

where \( D \) is the discriminant of \( K/k \) and \( ( \cdot, \cdot )_v \) is the local Hilbert symbol.

The collection of local spaces \( (\cdots, M'_v, \cdots) \) comes from a global Hermitian space \( M' \) if and only if the image of \( (\cdots, d(M'_v), \cdots) \) is trivial in the group

\[
\mathbb{A}^x/(\mathbb{N}A^x_K \cdot k^x) \cong \text{Gal}(K/k).
\]

Since \( \prod_v (d(M), D)_v = 1 \) by global reciprocity, a global space \( M' \) exists precisely when

\[
\prod_v \chi_v(-1, 1) = 1.
\]

But our formulae for \( \chi_v \) evaluated at \((-1, 1)\) give,

\[
\chi_v(-1, 1) = \epsilon(\pi_v, V)\det_v(V_1 \otimes V_2)(e)e_{K_v}(-1)\frac{1}{2}\dim(V_1 \otimes V_2).
\]

Since

\[
\prod_v \det_v(V_1 \otimes V_2)(e) = \prod_v e_{K_v}(-1) = 1,
\]

by global reciprocity, we see that a global space \( M' \) exists with localizations \( M'_v \) if and only if

\[
\prod_v \epsilon(\pi_v, V) = \epsilon(\pi, V) = 1.
\]

In this case, the subspace \( M'_0 \) also exists globally, and is characterized by \( M'/M'_0 \cong M/M_0 \).

Hence the group \( H' \rightarrow G' \) exists over \( k \) when \( \epsilon(\pi, V) = 1 \). To determine if the representation \( \pi' \) appears in the discrete spectrum, we use a conjectural multiplicity formula of Langlands and Arthur. Let \( \varphi : L_k \rightarrow L\Gamma \) be the global Langlands parameter of the tempered automorphic representation \( \pi \), and let \( A_{\varphi_v} \) be the component group of its centralizer. This is again an elementary abelian 2-group, determined by the irreducible submodules \( V_i \) of \( V_1 \otimes V_2 \), which maps to the local groups \( A_{\varphi_v} \) for all places \( v \). The representation
\( \pi' = \otimes \pi'_v \) should appear in the discrete spectrum with multiplicity 1 if and only if the character \( \chi : A_\varphi \to \langle \pm 1 \rangle \) defined by

\[
\chi(a) = \prod_v \chi_v(a)
\]

is trivial. When \( \dim V^{a=-1} \), we can use the formula for \( \chi_v(a) \) and global reciprocity to show that this is equivalent to the condition:

\[
\prod_v \epsilon(\pi_v, V^{a=-1}) = \epsilon(\pi, V^{a=-1}) = 1.
\]

If \( \epsilon(\pi, V^{a=-1}) = 1 \) for all \( a \) in the centralizer of the parameter, there is no obvious reason for the global \( L \)-function \( L(\pi, V, s) \) to vanish at the central critical point \( s = \frac{1}{2} \). We then make the main global conjecture.

**Conjecture 12.1.** Assume that \( \epsilon(\pi, V^{a=-1}) = 1 \) for all \( a \) in \( A_\varphi \). Then the adelic representation \( \pi' \) appears with multiplicity 1 in the discrete spectrum of \( G' \), and the period integrals

\[
f \to \int_{H'(k) \backslash H'(\mathbb{A})} f(h) \, dh
\]

of functions \( f \) in \( \pi' \) give a non-zero \( H'(\mathbb{A}) \)-invariant linear form if and only if \( L(\pi, V, \frac{1}{2}) \neq 0 \).

The recent work of Ginzburg-Jiang-Rallis [GJR] gives definitive progress towards this global conjecture.

**Notes**

(1) As in [Ichino-Ikeda], one can further conjecture an exact formula relating these periods and the central critical value.

(2) When \( \epsilon(\pi, V) = -1 \), the collection of local Hermitian spaces \( M'_0, v \subset M'_v \) does not arise from a global space. However, in the situation where \( k \) is a totally real number field and the local spaces \( M'_v \) are definite at all real places of \( k \), one should be able to use the arithmetic geometry of unitary Shimura varieties of rank 1 to study the central critical derivative \( L'(\pi, V, \frac{1}{2}) \) using the framework of [G].

Our conjectures relate central critical \( L \)-values of symplectic representations (i.e., for which the exterior-square \( L \)-function has a pole) to certain period integrals. It seems reasonable to expect that these are the only \( L \)-functions which can vanish at the center of the critical strip. We state it explicitly in the following.

**Conjecture 12.2.** Let \( \Pi \) be a cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}_\mathbb{Q}) \) with unitary central character. Suppose

\[
L\left(\frac{1}{2}, \Pi\right) = 0.
\]
Then, \[ \Pi \cong \Pi^\vee, \]
and \( \Pi \) is a symplectic automorphic representation, i.e.,
\[ L(s, \bigwedge^2 \Pi) \]
has a pole at \( s = 1 \).

Remark 1. Considerations of Artin representations of the Galois group suggest that it is essential to have \( \mathbb{Q} \) in the above conjecture, and which is of course no loss of generality when dealing with \( L \)-functions. We have not found any explicit reference to such a precise conjecture even for Dirichlet characters besides the well-known conjecture about quadratic characters of \( \mathbb{Q} \), which is, \( L(\frac{1}{2}, \omega) \neq 0 \) for any character \( \omega : \mathbb{A}_\mathbb{Q}^\times / \mathbb{Q}^\times \to \pm 1 \).

Part 2. EVIDENCE

In this part of the paper, we verify our conjectures about branching laws from \( U(2) \) to \( U(1) \), as well as for those representations of \( U(3) \) restricted to \( U(2) \) whose parameter restricted to \( K \) becomes reducible. We should add that even for these small groups, since there is no natural labelling of representations in an \( L \)-packet through characters of the corresponding component group, our verification will be for the form of the conjectures in 6.1 and 6.2, and not the more precise one formulated in sections 7, 8, 9.

13. \( U(1,1) \)

The group \( U(1,1) \) is a small variation on \( GL_2 \), and our conjectures 6.1 and 6.2 reduce to known results in this case. We discuss this here.

Let \( U(1,1) \) be the unitary group defined using the Hermitian form
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]
If \( GU(1,1) \) denotes the unitary similitude group, then it is easy to see that
\[ GU(1,1) \cong [GL_2(k) \times K^\times] / \Delta k^\times, \]
where \( \Delta k^\times \) sits inside \( GL_2(k) \) as the scalar matrices, and inside \( K^\times \) as \( t \to t^{-1} \). The similitude character on \( GU(1,1) \) in its identification as \( [GL_2(k) \times K^\times] / \Delta k^\times \) is the determinant character on \( GL_2(k) \), and the norm on \( K^\times \). Define \( GL_2^+(k) \) be the subgroup of \( GL_2(k) \) consisting of elements of \( GL_2(k) \) with determinant in \( NK^\times \). It follows that \( U(1,1) \) is contained inside \( G = [GL_2^+(k) \times K^\times] / \Delta k^\times \). Clearly \( U(1,1) \) and \( K^\times \) inside \( G \) commute, and generate \( G \) with \( U(1,1) \cap K^\times = U(1) \). Thus a representation \( \pi \) of \( U(1,1) \) can be extended to a representation of \( G \) by simply extending the central character \( \omega_\pi \) of \( \pi \) from \( K^\times = U(1) \) to a character \( \chi \) of \( K^\times \). The representation of \( G \) restricted to \( GL_2(k)^+ \) is irreducible with central character \( \chi|_{K^\times} \).
The group $U(1,1)$ being the subgroup of $GU(1,1)$ consisting of elements of similitude factor 1, sits in the following exact sequence

$$1 \rightarrow U(1,1) \rightarrow [GL_2^+(k) \times K^\times]/\Delta k^\times \rightarrow K^\times \rightarrow 1.$$

We have the natural embedding of $U(1)$ inside $U(1,1)$ given through its action on a one dimensional subspace of the two dimensional Hermitian space on which $U(1,1)$ operates, such that the action of $U(1)$ on the orthogonal complement (which is a one dimensional Hermitian space with discriminant one) is trivial. Thinking of $U(1)$ as $K^\times/k^\times$, the embedding of $U(1)$ inside $U(1,1)$ is given by

$$K^\times/k^\times \hookrightarrow [GL_2^+(k) \times K^\times]/\Delta k^\times$$

where $K^\times$ embeds in $GL_2(k)^+$ through one of its embeddings (there are possibly two of them!), and the mapping from $K^\times$ to $K^\times$ is $t \mapsto t^{-1}$.

It thus follows that for a character $\mu$ of $K^\times/k^\times$ to appear in the restriction of a representation $\pi \times \chi$ of $[GL_2^+(k) \times K^\times]/\Delta k^\times$, it is necessary and sufficient that the character $\mu\bar{\chi}$ of $K^\times$ appears in the representation $\pi$ of $GL_2(k)$. This is exactly the refinement of the theorem of Saito and Tunnell considered by the third author in [P2] which we state now. In the statement of the theorem below, we will not consider those representations of $GL_2(k)$ which remain irreducible when restricted to $GL_2^+(k)$, being known from results due to Saito and Tunnell for $GL_2(k)$.

**Theorem 13.1.** A representation of $GL_2(k)$ decomposes into two irreducible components when restricted to $GL_2^+(k)$ if and only if its Langlands parameter is induced from a character, say $\alpha$, of $K^\times$. Suppose that this is the case. The two representations of $GL_2^+(k)$ so obtained can be indexed as $\pi^+$ and $\pi^-$ in such a way that a character $\beta$ of $K^\times$ with $\alpha|_{k^\times} = \omega_{K/k}|_{k^\times}$ appears in $\pi^+$ if and only if $\epsilon(\alpha\beta^{-1}, \psi) = \epsilon(\bar{\alpha}\beta^{-1}, \psi) = 1$, and appears in $\pi^-$ if and only if $\epsilon(\alpha\beta^{-1}, \psi) = \epsilon(\bar{\alpha}\beta^{-1}, \psi) = -1$; here $\psi$ is a fixed additive character of $K$ which is trivial on $k$, changing which will change the ordering of $\pi^+$ and $\pi^-$. There is a similar statement for $D^{++}$.

13.1. **Parameter for $U(1,1)$**. For translating the above theorem about $GL_2^+(k)$ to $U(1,1)$, one needs to associate $L$-parameters to representations of $U(1,1)$. Assume therefore that $\pi$ is a representation of $U(1,1)$ obtained by restriction of a representation $\pi \times \chi$ of $[GL_2^+(k) \times K^\times]/\Delta k^\times$. Then the $L$-parameter of $\pi$ restricted to $K$ is $\sigma_{\pi}|_K \otimes \bar{\chi}^{-1}$ where $\sigma_{\pi}$ is the parameter for $\pi$.

In the notation of the previous theorem, $\sigma_{\pi}|_K = \alpha \oplus \bar{\alpha}$, and therefore the parameter of the representation of $U(1,1)$ restricted to $K$ is $\alpha\bar{\chi}^{-1} \oplus \bar{\alpha}\chi^{-1}$ where we recall that $(\alpha\chi^{-1})|_{k^\times} = \omega_{K/k}$. Therefore conjectures 6.1 and 6.2 predict that a character $\mu$ of $K^\times/k^\times$ appears in $\pi$ if and only if $\epsilon(\alpha\mu^{-1}, \psi) = \pm 1$, and $\epsilon(\bar{\alpha}\chi^{-1}\mu^{-1}) = \pm 1$, take fixed values. On the other hand as observed before, $\mu$ appears in $\pi$ if and only if the character $\mu\bar{\chi}$ appears in $\pi$. Thus the conclusion of Theorem 13.1 is exactly what the conjectures 6.1 and 6.2 predict. We note for purposes here, as well as for later use, that for a character $\mu$ of $U(1) = K^\times/k^\times$, its base change to $K$, i.e., to $K^\times$ is the character $\mu$ itself.
14. Trilinear forms for \( U(2) \)

Given \( \pi_1, \pi_2, \pi_3 \), three irreducible admissible representations of \( G = \text{GL}_2(k)^+ \) or \( U(2) \), with the product of their central characters trivial, we calculate the dimension of the space of trilinear forms

\[
\dim \text{Hom}_G[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}].
\]

These results when combined with Seesaw duality in theta correspondence will translate into branching laws from \( U(3) \) to \( U(2) \) in the next section.

Let \( G \) be a subgroup of \( \text{GL}_2(k) \) containing \( \text{SL}_2(k) \). The group \( G \) is uniquely determined by the subgroup \( k_G^x \) of \( k^x \) consisting of determinants of elements of \( G \). It thus makes sense to define a corresponding subgroup, say \( G_D \), inside \( D^x \) containing \( \text{SL}_1(D) \). Restricting representations of \( \text{GL}_2(k) \) to \( G \), or of \( D^x \) to \( G_D \), one gets a notion of \( L \)-packet of representations of \( G \), and of \( G_D \). Representations of \( \text{GL}_2(k) \) restrict to \( G_D \) with multiplicity 1, but this need not be the case for representations of \( D^x \) restricted to \( G_D \). For a representation \( \pi' \) of \( G_D \), let \( m(\pi') \) denote the multiplicity with which it appears in an irreducible representation of \( D^x \) when restricted to \( G_D \).

**Theorem 14.1.** Let \( \hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3 \) be three irreducible admissible representations of \( \text{GL}_2(k) \) with the product of their central characters equal to 1. Let \( \hat{\pi}'_1, \hat{\pi}'_2, \hat{\pi}'_3 \) be the corresponding three irreducible representations of \( D^x \) associated to \( \hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3 \) by the Jacquet-Langlands correspondence. (If \( \hat{\pi} \) is not essentially square-integrable representation of \( \text{GL}_2(k) \), we let \( \hat{\pi}' = 0 \).) Then,

\[
\sum_{\pi_1, \pi_2, \pi_3} \dim \text{Hom}_G[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}]
+ m(\pi'_1)m(\pi'_2)m(\pi'_3) \sum_{\pi_1', \pi_2', \pi_3'} \dim \text{Hom}_{G_D}[\pi_1' \otimes \pi_2' \otimes \pi_3', \mathbb{C}] = \#(k^x / k^x k_G^x),
\]

where the sum is taken over irreducible representations \( \pi_1, \pi_2, \pi_3 \) of \( G \) (or, \( \pi_1, \pi_2, \pi_3 \) of \( G_D \)) which are contained in the representations \( \hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3 \) of \( \text{GL}_2(k) \), or representations \( \hat{\pi}'_1, \hat{\pi}'_2, \hat{\pi}'_3 \) of \( D^x \).

**Proof.** Clearly,

\[
\text{Hom}_G[\hat{\pi}_1 \otimes \hat{\pi}_2 \otimes \hat{\pi}_3, \mathbb{C}] \cong \sum_{\chi : k^x / k^x G^x \to \mathbb{Z}/2} \text{Hom}_{\text{GL}_2(k)}[\hat{\pi}_1 \otimes \hat{\pi}_2 \otimes \hat{\pi}_3, \mathbb{C}_\chi],
\]

where \( \chi \)’s are characters of \( \text{GL}_2(k) \) trivial on \( G \), and \( \mathbb{C}_\chi \) denotes the 1-dimensional representation of \( \text{GL}_2(k) \) on which it operates by the character \( \chi \); by considerations of central character, \( \chi \)’s are of order \( \leq 2 \). Similarly,

\[
\text{Hom}_{G_D}[\hat{\pi}'_1 \otimes \hat{\pi}'_2 \otimes \hat{\pi}'_3, \mathbb{C}] \cong \sum_{\chi : k^x / k^x G^x \to \mathbb{Z}/2} \text{Hom}_{D^x}[\hat{\pi}'_1 \otimes \hat{\pi}'_2 \otimes \hat{\pi}'_3, \mathbb{C}_\chi].
\]

But by [P],
\[ \dim \text{Hom}_{\text{GL}_2(k)}[\hat{\pi}_1 \otimes \hat{\pi}_2 \otimes \hat{\pi}_3, \mathbb{C}_\chi] + \dim \text{Hom}_{\text{G}_D}[\hat{\pi}'_1 \otimes \hat{\pi}'_2 \otimes \hat{\pi}'_3, \mathbb{C}_\chi] = 1 \]

for all characters \( \chi \) of order \( \leq 2 \) (by absorbing \( \chi \) in one of the \( \pi'_i \)'s).

Thus by adding up the contribution of the various \( \chi \)'s, we get the conclusion of the theorem.

\[ \square \]

**Corollary 14.2.** Let \( G = \text{GL}_2^+(k) \). Then as \( m(\pi') = 1 \) for all irreducible representations \( \pi' \) of \( D^\times \), one has,

\[ \sum_{\pi_1, \pi_2, \pi_3} \dim \text{Hom}_G[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}] + \sum_{\pi'_1, \pi'_2, \pi'_3} \dim \text{Hom}_{\text{G}_D}[\pi'_1 \otimes \pi'_2 \otimes \pi'_3, \mathbb{C}] = 2, \]

where the sum is taken over irreducible representations \( \pi_1, \pi_2, \pi_3 \) of \( G \) (or, \( \pi'_1, \pi'_2, \pi'_3 \) of \( \text{G}_D \)) which are contained in the representations \( \hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3 \) of \( \text{GL}_2(k) \), or representations \( \hat{\pi}'_1, \hat{\pi}'_2, \hat{\pi}'_3 \) of \( D^\times \).

The group \( \text{U}(1, 1) \) sits in the following exact sequence

\[ 1 \rightarrow \text{U}(1, 1) \rightarrow [\text{GL}_2(k) \times K^\times]/\Delta k^\times \rightarrow k^\times \rightarrow 1. \]

Any irreducible representation of \( \text{U}(1, 1) \) is contained in the restriction of a representation \( \pi \times \chi_0 \) of \([\text{GL}_2(k) \times K^\times]/\Delta k^\times \) where \( \pi \) is a representation of \( \text{GL}_2(k) \) and \( \chi_0 \) is a character of \( K^\times \) such that if \( \omega_\pi \) denotes the central character of \( \pi \) then \( \omega_\pi = \chi_0|_{k^\times} \); in fact the restriction of the \( \pi \times \chi_0 \) of \([\text{GL}_2(k) \times K^\times]/\Delta k^\times \) to \( \text{U}(1, 1) \) gives an \( L \)-packet of representations on \( \text{U}(1, 1) \). By an analysis done exactly as in theorem 15.1, we get the following:

**Corollary 14.3.** Let \( \text{U}'(2) \) and \( \text{U}''(2) \) be the two unitary groups corresponding to the 2 Hermitian space of dimension 2. Then

\[ \sum_{\pi'_1, \pi'_2, \pi'_3} \dim \text{Hom}_{\text{U}'(2)}[\pi'_1 \otimes \pi'_2 \otimes \pi'_3, \mathbb{C}] + \sum_{\pi''_1, \pi''_2, \pi''_3} \dim \text{Hom}_{\text{U}''(2)}[\pi''_1 \otimes \pi''_2 \otimes \pi''_3, \mathbb{C}] = 2, \]

where the sum is taken over irreducible representations \( \pi'_1, \pi'_2, \pi'_3 \) of \( \text{U}'(2) \), and \( \pi''_1, \pi''_2, \pi''_3 \) of \( \text{U}''(2) \) which belong to the same Vogan packet of representations with the product of their central characters trivial.

**Corollary 14.4.** (Multiplicity 1) For \( G = \text{GL}_2(k)^+ \) or \( \text{U}(2) \), let \( \pi_1, \pi_2, \pi_3 \) be 3 irreducible admissible representations of \( G \), such that for one of the representations, say \( \pi_1 \), its \( L \)-packet has more than 1 element (so 2). Then

\[ \dim \text{Hom}_G[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}] \leq 1. \]

**Proof.** If \( \text{Hom}_G[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}] \neq 0 \), then so is \( \text{Hom}_G[\pi'_1 \otimes \pi'_2 \otimes \pi'_3, \mathbb{C}] \), where \( \pi'_i \) denotes the conjugate of \( \pi_i \) by an element of \( \text{GL}_2(k) \) which is not in \( \text{GL}_2(k)^+ \). Since the sum of the dimensions of \( \text{Hom}_G[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}] \) and \( \text{Hom}_G[\pi'_1 \otimes \pi'_2 \otimes \pi'_3, \mathbb{C}] \), is bounded by 2, each one is bounded by 1.

\[ \square \]

Although the next two corollaries play no role in this paper, we point them out anyway.
Corollary 14.5. (Failure of multiplicity 1) If \( \hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3 \) are three irreducible admissible representations of \( \text{GL}_2(k) \) with the product of their central characters trivial, and such that each of the \( \hat{\pi}_i \)'s remain irreducible when restricted to \( \text{GL}_2(k)^+ \), and one of them is a principal series representation, then
\[
\dim \text{Hom}_{\text{GL}_2^+(k)}[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}] = 2.
\]

Corollary 14.6. (High multiplicity for \( \text{SL}_2 \)) Since \( k^\times / k^\times 2 \) is a 2-group whose cardinality can be made arbitrarily large by choosing \( k \) appropriately, and since the \( L \)-packet of representations of \( \text{SL}_2(k) \) is bounded by 4, it follows that the multiplicity,
\[
m(\pi_1, \pi_2, \pi_3) = \dim \text{Hom}_{\text{SL}_2(k)}[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}]
\]
can be made arbitrarily large.

15. Using the theta correspondence for \( \text{U}(2, 1) \)

In this section we will use methods of theta correspondence to prove our conjectures for certain representations of \( \text{U}(2, 1) \) in both real and the \( p \)-adic case. These are the representations of \( \text{U}(2, 1) \) which can be obtained as a theta lift from smaller unitary groups. This method in particular covers all the discrete series representations of \( \text{U}(2, 1) \) over the reals. It should be noted here that all our conjectures about restriction of a representation of \( \text{U}(n) \) to \( \text{U}(n-1) \) are invariant under simultaneous twisting of the representation of \( \text{U}(n) \) and of \( \text{U}(n-1) \) by the same character of \( \text{U}(1) \). Therefore, although the parameters of the representations of \( \text{U}(2, 1) \) obtained by theta lifting from \( \text{U}(2) \) appears to be smaller than the set of parameters of \( \text{U}(2, 1) \) which become reducible when restricted to \( K \), this will not be an issue for us.

Here is the main theorem about \( \text{U}(2, 1) \) that we shall be able to prove.

Theorem 15.1. Let \( \sigma : W'_k \to L \text{U}(3) \) be the parameter of an irreducible representation \( \pi \) of a unitary group \( \text{U}(3) \) in 3-variables over a local field. Let \( K \) be the quadratic field defining the unitary group \( \text{U}(3) \), and let \( \sigma_K : W'_K \to \text{GL}(3, \mathbb{C}) \) be the restriction of the parameter \( \sigma \) to \( K \). Assume that there is a character \( \chi : K^\times / k^\times \to \mathbb{C}^\times \) such that
\[
\sigma_K = \sigma_1 \oplus \chi.
\]

(1) If \( \sigma_1 \) is an irreducible 2-dimensional representation of \( W'_k \), the extended \( L \)-packet associated to \( \sigma \) has 4 elements, 2 of which lie on this \( U(3) \), and 2 lie on the other (isomorphic) copy of \( U(3) \) defined by the non-isomorphic Hermitian form. Let \( \sigma_2 : W'_k \to L \text{U}(2) \) be the parameter of an irreducible admissible representation \( \pi_2 \) of \( \text{U}(2) \) (which is any unitary group in 2 variables defined by a rank 2 Hermitian form over \( K \) contained in the rank 3 Hermitian form defining \( \text{U}(3) \)). Then if \( \text{Hom}_{\text{U}(2)}[\pi \otimes \pi_2, \mathbb{C}] \neq 0 \), \( \epsilon(\sigma_1 \otimes \sigma_{2, K}) \) and \( \epsilon(\chi \otimes \sigma_{2, K}) \) take values in \( \pm 1 \) independent of \( \pi_2 \) (as long as \( \text{Hom}_{\text{U}(2)}[\pi \otimes \pi_2, \mathbb{C}] \neq 0 \)). By varying \( \pi \) in the extended packet (with 4 elements), the characters so obtained on \( (\mathbb{Z}/2)^2 \) gives a bijection of the members of the \( L \)-packet of \( \pi \) with characters of \( (\mathbb{Z}/2)^2 \).
(2) If $\sigma_1$ is a sum of 2 characters, say $\sigma_1 = \alpha \oplus \beta$, the extended L-packet associated to $\sigma$ has 8 elements, 4 of which lie on this $U(3)$, and 4 lie on the other (isomorphic) copy of $U(3)$ defined by the non-isomorphic Hermitian form. Let $\sigma_2 : W_2^\prime \to U(2)$ be the parameter of an irreducible admissible representation $\pi_2$ of $U(2)$ (which is any unitary group in 2 variables defined by a rank 2 Hermitian form over $K$ contained in the rank 3 Hermitian form defining $U(3)$). Then if $\text{Hom}_{U(2)}[\pi \otimes \pi_2, \mathbb{C}] \neq 0$, $\epsilon(\alpha \otimes \sigma_2, K)$, $\epsilon(\beta \otimes \sigma_2, K)$, $\epsilon(\chi \otimes \sigma_2, K)$ take values in $\pm 1$ independent of $\pi_2$ (as long as $\text{Hom}_{U(2)}[\pi \otimes \pi_2, \mathbb{C}] \neq 0$).

Our proof about restriction from $U(V)$ to $U(W)$ will be based on the following seesaw diagram where $W$ is a codimension one subspace of $V$ such that $V = W \oplus K$ as Hermitian spaces:

$$
\begin{array}{ccc}
U(2, 1) & U(W') \times U(W') & U(W) \times U(1) \\
& \Delta U(W').
\end{array}
$$

It follows from this seesaw diagram that the branching from $U(V)$ to $U(W)$ will be based on the following two informations.

1. Explicit theta lifting between unitary groups in 2 variables with those in 1 and 3 variables.
2. Tensor product of two irreducible representations of $U(W')$ where $W'$ is a two dimensional Hermitian space.

Before we discuss the theta lifting between unitary groups, let us remind ourselves that the theta correspondence between unitary groups $U(W_1)$ and $U(W_2)$ depends crucially on fixing characters $\chi_1$ and $\chi_2$ on $K^\times$ such that $\chi_1|_{k^\times} = \omega_{K/k}^\dim W_2$, and $\chi_2|_{k^\times} = \omega_{K/k}^\dim W_1$, which are used to fix a lifting of $U(W_1) \times U(W_2)$ to the metaplectic group corresponding to the Hermitian space $W_1 \otimes_K W_2$. We will assume fixing the characters $(\chi_1, \chi_2) = (\mu^\dim W_2, \mu^\dim W_1)$ where the character $\mu$ of $K^\times$ (with restriction to $k^\times$ equal to $\omega_{K/k}$) is chosen as before. The choice of these characters has the advantage that for the corresponding liftings of $U(W_1)$ and $U(W_2)$, the centers of $U(W_1)$ and $U(W_2)$ get identified to each other, and as a result, the theta lifting preserves the central characters, cf. [HKS, Corollary A.8]. Furthermore, the seesaw diagram,

$$
\begin{array}{ccc}
U(W_1 \oplus W_2) & U(V) \times U(V) & U(W_1) \times U(W_2) \\
& \Delta U(V).
\end{array}
$$
which exists at the level of symplectic groups, works exactly the same way in the metaplectic covering groups, in which we will use the characters \((\mu_{\dim V}, \mu_{\dim W_1})\) for the dual pair \((U(W_1), U(V))\), \((\mu_{\dim V}, \mu_{\dim W_1 + \dim W_2})\) for the dual pair \((U(W_1 \oplus W_2), U(V))\) etc.

Suppose that \(\pi_1\) is an irreducible admissible representation of \(U(W_1)\) and \(\theta(\pi_1) = \pi_2\) is its theta lift to \(U(W_2)\) which we assume is nonzero. Assume that \(\sigma_1\) and \(\sigma_2\) are the Langlands parameters of \(\pi_1\) and \(\pi_2\) respectively, and that \(\sigma_{1,K}\) and \(\sigma_{2,K}\) denote their restrictions of \(W_K\). We will then use the following about theta correspondence:

1. If \(\dim W_1 = \dim W_2\), then \(\sigma_{1,K} = \sigma_{2,K}\).
2. If \(\dim W_1 = \dim W_2 + 1\), then \(\sigma_{1,K} = \mu \sigma_{2,K} \oplus \mu^{-\dim W_2}\).

Although one would expect these in general, we will need to use these relations for the Langlands parameters of representations related by theta liftings only for the pairs \((U(1), U(2))\), and \((U(2), U(3))\) which are well known, such as in [GRS] for the second pair. One expects furthermore, that in the pair \((U(1), U(2))\) with \(\dim W_1 = \dim W_2 + 1\), as one varies representations in a Vogan packet on \(U(W_2)\), one gets a Vogan packet on the larger unitary group \(U(W_1)\). Again, this is known when \(\dim W_2\) is 1 or 2.

Given a character \(\nu\) of \(K^1 \subset K^\times\), let \(\tilde{\nu}\) be an extension of \(\nu\) to \(K^\times\). Define \(\chi_\nu(z) = \nu(z/\bar{z}) = \tilde{\nu}(z/\bar{z})\). Observe that \(z \to z/\bar{z}\) being the norm mapping for \(U(1)\), \(\chi_\nu(z) = \nu(z/\bar{z})\) is the base change of \(\nu\).

With this notation, we can write the theta lifting of the character \(\nu\) of \(U(1)\) to be a representation of \(U(1,1)\) which is contained in the representation
\[
(3) \text{Ind}(\mu \tilde{\nu}) \otimes \tilde{\nu}(x) \text{ of } [GL_2(k) \times K^\times]/\Delta k^\times.
\]

Notice that this representation of \([GL_2(k) \times K^\times]/\Delta k^\times\), from the recipe in section 13.1, the Langlands parameter of the representation of \(U(1,1)\) after base change to \(K\) is:
\[
(4) \text{Ind}(\mu \tilde{\nu})|_K \otimes \tilde{\nu}^{-1}(\bar{x}) = [\mu \tilde{\nu} \oplus \bar{\mu} \tilde{\nu}] \otimes \tilde{\nu}(\bar{x})^{-1} = \mu \chi_\nu(x) \oplus \mu^{-1},
\]
which matches with the parameter as given in (2) (after noticing that the base change of the character \(\nu\) of \(U(1)\) is the character \(\chi_\nu(x) = \nu(x/\bar{x})\) of \(K^\times\)).

Recall from the previous section that \(U(1,1)\) is a subgroup of \(G = [GL_2^+(k) \times K^\times]/\Delta k^\times\) such that any irreducible representation of \(U(1,1)\) is the restriction of an irreducible representation of \(G\), and further, an irreducible representation of \(G\) remains irreducible when restricted to \(U(1,1)\). It follows that the calculation of the tensor product of irreducible representations of \(U(1,1)\) is essentially the same as that of \(G\). As \(G = [GL_2^+(k) \times K^\times]/\Delta k^\times\), clearly calculation of tensor product of irreducible representations of \(G\) is the same as that of \(GL_2^+(k)\), something which has been essentially done by the 3rd author, cf. [P5]. We will content ourselves by stating the result in just one case.

**Theorem 15.2.** Let \(\pi_1, \pi_2\) be two irreducible admissible representations of \(GL_2^+(k)\), and let \(\pi_3\) be one of \(GL_2(k)\) such that the product of the central characters \(\omega_{\pi_1} \omega_{\pi_2} \omega_{\pi_3} = 1\). Assume that the Langlands parameters \(\sigma_i\) of the representations \(\pi_i\) are obtained from characters \(\alpha_i\) of \(K^\times\) for \(i = 1, 2\). Then if
\[
\text{Hom}_{GL_2^+(k)}[\pi_1 \otimes \pi_2 \otimes \pi_3, C] \neq 0,
\]
\[ \epsilon(\text{Ind}(\alpha_1 \alpha_2) \otimes \sigma_3) \text{ and } \epsilon(\text{Ind}(\alpha_1 \alpha_2) \otimes \sigma_3) \text{ take values in } \pm 1, \text{ independent of } \pi_3 \text{ as long as} \]

\[ \text{Hom}_{\GL_2^+(k)}[\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}] \neq 0. \]

Similarly for \( D^\times \). The character of \( (\mathbb{Z}/2)^2 \) so obtained (by these two epsilon values) identifies the extended \( L \)-packet of \( \GL_2^+(k) \times \GL_2^+(k) \) (and \( D^\times \times D^\times \)) containing \( \pi_1 \times \pi_2 \) in which the representations \( \pi_1 \times \pi_2 \) and \( \pi'_1 \times \pi'_2 \) are identified where \( \pi'_1 \) (resp. \( \pi'_2 \)) is the other member of the \( L \)-packet containing \( \pi_1 \) (resp. \( \pi_2 \)).

15.1. Completing the branching from \( U(2, 1) \) to \( U(2) \). We look at the seesaw diagram

\[
\begin{array}{ccc}
U(2, 1) & \rightarrow & U(W') \times U(W') \\
\downarrow & & \downarrow \\
U(W) \times U(1) & \rightarrow & \Delta U(W')
\end{array}
\]

starting with a representation \( \pi_1 \) of \( \Delta U(W') \) with Langlands parameter restricted to \( K \) equal to \( \sigma_1 \), and a representation \( \pi_2 \times \nu \) of \( U(W) \times U(1) \) with Langlands parameter restricted to \( K \) equal to \( \sigma_2 \times \nu \). The seesaw identity gives us,

\[ \text{Hom}_{U(W') \times U(1)}[\theta(\pi_1), \pi_2 \times \nu] = \text{Hom}_{U(\Delta W')}[\theta(\pi_2) \times \theta(\nu), \pi_1]. \]

Here \( U(1) \) is the unitary group of a Hermitian form in 1 variable, and \( U(W), U(\Delta W') \) are the unitary groups of Hermitian forms in 2 variables. Noting that the direct sum of Hermitian spaces of dimension 1 and 2 creates a Hermitian space of dimension 3, and that this direct sum construction is a two-to-one mapping onto Hermitian spaces of dimension 3, multiplicity 2 theorem about triple products of representations of \( U(2) \) (Corollary 14.3) translates to multiplicity 1 theorem for the restriction of Vogan \( L \)-packet on \( U(3) \) to a Vogan \( L \)-packet on \( U(2) \).

We now carry out the comparison of relevant epsilon factors. The Langlands parameter of \( \theta(\pi_1) \) restricted to \( K \) is \( \sigma_3 = \mu \sigma_1 \oplus \mu^{-2} \), therefore the branching from \( U(3) \) to \( U(2) \) depends on \( \epsilon([\mu \sigma_1 \oplus \mu^{-2}] \otimes \sigma_1') \), and therefore on the epsilon factors (of representations over \( K \)):

\[ (15.1) \quad \epsilon(\mu \sigma_1 \otimes \sigma_1') \text{, and} \]

\[ (15.2) \quad \epsilon(\mu^{-2} \sigma_1'). \]

On the other hand, assuming that the representations \( \pi_1 \) and \( \pi_2 \) of \( U(1, 1) \) arise from representations of \( \GL(2,k) \) with Langlands parameters \( \tau_1 \) and \( \tau_2 \) (and with base change parameters \( \sigma_1 \) and \( \sigma_2 \)), the non-vanishing of \( \text{Hom}_{U(2)}[\theta(\pi_2) \times \theta(\nu), \pi_1] \) is controlled by the triple product epsilon factor :
\[ \epsilon(\tau_1^\vee \otimes \tau_2 \otimes \text{Ind}(\mu\tilde{\nu})) = \epsilon(BC(\tau_1)^\vee \otimes BC(\tau_2) \otimes \mu\tilde{\nu}). \]

(Here we have used the notation, \( BC(V) \) to denote the restriction of a representation \( V \) of \( W_K \) to \( W_K \), and the fact that \( \epsilon(V \otimes \text{Ind}\alpha) = \epsilon(V|_K \otimes \alpha) \) whenever \( 4|\dim V. \) Let \( \omega_1 \) and \( \omega_2 \) be the central characters of the representations \( \pi_1 \) and \( \pi_2 \) of \( U(\Delta W') \) and \( U(W) \) respectively. So, \( \omega_1 \) and \( \omega_2 \) are characters of \( U(1) = K^\times/k^\times \). Comparing central character of the representation \( \theta(\pi_1) \) of \( U(2, 1) \) with that of the representation \( \pi_2 \times \nu \) of \( U(2) \times U(1) \), we must have

\[ \omega_1 = \omega_2 \cdot \nu, \text{ or, } \nu = \omega_1\omega_2^{-1}. \]

Let \( \tilde{\omega}_1, \tilde{\omega}_2 \) and \( \tilde{\nu} \) be character of \( K^\times \) extending the characters \( \omega_1, \omega_2, \nu \) of \( K^1 = U(1) \). We assume that these extensions are so chosen that,

\[ \tilde{\nu} = \tilde{\omega}_1\tilde{\omega}_2^{-1}. \]

Recall that for a representation \( \pi \) of \( U(1, 1) \) obtained by restriction of a representation \( \pi \times \chi \) of \( [GL^+_1(k) \times K^\times]/\Delta k^\times \), the \( L \)-parameter of \( \pi \) restricted to \( K \) is \( \sigma_{\pi}|_K \otimes \chi^{-1} \) where \( \sigma_{\pi} \) is the parameter for \( \pi \). Therefore,

\[ BC(\tau_1) = \sigma_1 \otimes \tilde{\omega}_1, \text{ and } BC(\tau_2) = \sigma_2 \otimes \tilde{\omega}_2. \]

Therefore,

\[ BC(\tau_1)^\vee \otimes BC(\tau_2) \otimes \tilde{\nu} = \sigma_1^\vee \otimes \sigma_2 \otimes (\tilde{\omega}_1^{-1}\tilde{\omega}_2\tilde{\nu}) = \sigma_1^\vee \otimes \sigma_2. \]

Therefore,

\[ \epsilon(\tau_1^\vee \otimes \tau_2 \otimes \text{Ind}(\mu\tilde{\nu})) = \epsilon(BC(\tau_1)^\vee \otimes BC(\tau_2) \otimes \mu\tilde{\nu}) = \epsilon(\sigma_1^\vee \otimes \sigma_2 \otimes \mu) = \epsilon(\sigma_1 \otimes \sigma_2^\vee \otimes \mu^{-1}) = \epsilon(\sigma_1 \otimes \sigma_2^\vee \otimes \mu). \]

Here we have used the facts that \( \epsilon(V) = \epsilon(V^\vee) \) if \( \det V(-1) = 1 \), as is the case for us, and in such a case, \( \epsilon(V) = \epsilon(V^\sigma) \) where \( \sigma \) is the automorphism of \( K \) over \( k \).

Thus we get the same epsilon factor that we encountered in (15.1) for branching from \( U(3) \) to \( U(2) \). We note that by a theorem due to Harris-Kudla-Sweet in [HKS], the value of the other epsilon factor, \( \epsilon(\mu^{-2}\sigma^\vee_2) \) determines whether \( \theta(\pi_2) \) is zero or nonzero, therefore by the seesaw duality, the two epsilon factors in (15.1) and (15.2) together determine whether the representation \( \pi_2 \) of \( U(W) \) appears in the representation \( \theta(\pi_1) \) of \( U(2, 1) \).

16. Branching laws for \( GL(n, \mathbb{F}_q) \)

In this section we calculate the restriction of a representation of \( GL(n, \mathbb{F}_q) \) to \( GL(n - 1, \mathbb{F}_q) \) where \( GL(n - 1, \mathbb{F}_q) \) sits inside \( GL(n, \mathbb{F}_q) \) in the natural way as

\[ A \rightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \]
These branching laws are surely known in the literature, such as in the work of Thoma; however, we have preferred to give a different independent treatment.

We begin by recalling the notion of twisted Jacquet functor. For $P = MN$ any group such that $N$ is a normal subgroup of $P$ and $\varphi$ a character of $N$ which is left invariant under the inner-conjugation action of $M$ on $N$, the twisted Jacquet functor associates to any representation $V$ of $P$, the representation $V_{\varphi}$ which is the largest quotient of $V$ on which $N$ operates via the character $\varphi$; clearly $V_{\varphi}$ is a representation space for $M\varphi$, the subgroup of $M$ which operates trivially on $\varphi$. Let $E_{n-1}$ be the subgroup of $GL(n, \mathbb{F}_q)$ with last row equal to $(0,0,\cdots,0,1)$ and let $N_n$ be the group of upper triangular matrices in $GL(n, \mathbb{F}_q)$ with 1’s on the diagonal. We fix a nontrivial character $\psi_0$ of $k$ and let $\psi_n$ be the character of $N_n$, given by $\psi_n(u) = \psi_0(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$. For a representation $\pi$ of $GL(n, \mathbb{F}_q)$, let $\pi^i$ denote its $i$-th derivative which is a representation of $GL(n-i, \mathbb{F}_q)$. Recall that if $R_{n-i} = GL(n-i, \mathbb{F}_q) V_i$ is the subgroup of $GL(n, \mathbb{F}_q)$ consisting of

$$
\begin{pmatrix}
g & v \\
0 & z
\end{pmatrix}
$$

with $g \in GL(n-i, \mathbb{F}_q)$, $v \in M(n-i,i)$, $z \in N_i$, and if the character $\psi_i$ on $N_i$ is extended to $V_i$ by extending it trivially across $M(n-i,i)$, then $\pi^i = \pi \psi_i$.

If $\pi$ is an irreducible cuspidal representation of $GL(n, \mathbb{F}_q)$, then $\pi^i = \pi$ for $i = 0$, and $\pi^i = 1$, the trivial representation of $GL(0, \mathbb{F}_q)$ which is a group with 1 element; all the other derivatives of $\pi$ are 0.

The following proposition is from Bernstein-Zelevinsky [BZ], where it is done for non-Archimedean local fields, but works for finite fields as well.

**Proposition 16.1.** For $\pi_1$ a representation of $GL(n_1, \mathbb{F}_q)$ and $\pi_2$ of $GL(n_2, \mathbb{F}_q)$, we let $\pi_1 \times \pi_2$ denote the representation of $GL(n_1 + n_2, \mathbb{F}_q)$ induced from the corresponding representation of the parabolic with Levi subgroup $GL(n_1, \mathbb{F}_q) \times GL(n_2, \mathbb{F}_q)$. Then there is a composition series of the $k$-th derivative $(\pi_1 \times \pi_2)^k$ whose successive quotients are $\pi_1^i \times \pi_2^{k-i}$ for $i = 0, \cdots, k$.

Here is a generality from Bernstein and Zelevinsky [BZ].

**Proposition 16.2.** Any representation $\Sigma$ of $E_{n-1}$ has a natural filtration of $E = E_{n-1}$ modules $0 \subset \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n$ such that $\Sigma_{i+1}/\Sigma_i = \text{ind}_{R_i}^E(\Sigma^{n-i} \otimes \psi_{n-i})$ for $i = 0, \cdots, n$, where $R_i = GL(i, \mathbb{F}_q) \cdot V_{n-i}$ is the subgroup of $GL(n, \mathbb{F}_q)$ consisting of

$$
\begin{pmatrix}
g & v \\
0 & z
\end{pmatrix}
$$

with $g \in GL(i, \mathbb{F}_q)$, $v \in M(i, n-i)$, $z \in N_{n-i}$, and the character $\psi_{n-i}$ on $N_{n-i}$ is extended to $V_{n-i}$ by extending it trivially across $M(i, n-i)$.

The following corollary is clear.

**Corollary 16.3.** Let $n = n_1 + \cdots + n_r$ be a sum of positive integers, and let $\pi_i$ be an irreducible cuspidal representation of $GL(n_i, \mathbb{F}_q)$. Let $\Sigma = \pi_1 \times \cdots \times \pi_r$ be the
corresponding parabolically induced representation of \(GL(n, \mathbb{F}_q)\). Then the restriction
of \(\pi_1 \times \cdots \times \pi_r\) to \(GL(n-1, \mathbb{F}_q)\) is a sum of the following representations:

\[
\pi_{i_1} \times \pi_{i_2} \times \cdots \times \pi_{i_s} \times GG[n-1-(i_1+\cdots+i_s)]
\]

where \(1 \leq i_1 < i_2 < \cdots < i_s \leq r\) (the empty sequence is allowed), \(i_1 + \cdots + i_s < n\),
\(GG[m]\) denotes the Gelfand-Graev representation of \(GL(m, \mathbb{F}_q)\) which is the induced
representation of \(GL(m, \mathbb{F}_q)\) from a non-degenerate character of its group of upper-triangular
unipotent matrices; \(GG[1]\) is the regular representation of \(\mathbb{F}_q^\times\); \(GG[0]\) is the
trivial representation of the trivial group.

**Proof:** Since \(GL(n-1, \mathbb{F}_q)R_{n-1} = E_{n-1}\) for any \(i\), it follows from proposition 16.2 that
the restriction of \(\Sigma_{i+1}/\Sigma_i\) to \(GL(n-1, \mathbb{F}_q)\) is the representation \(\Sigma^{n-i} \times GG[n-i]\), where
\(GG[n-i]\) denotes the Gelfand-Graev representation of \(GL(n-i, \mathbb{F}_q)\) which is the induced
representation of \(GL(n-i, \mathbb{F}_q)\) from a non-degenerate character of its group of upper-triangular
unipotent matrices (where \(GG[1]\) is the regular representation of \(\mathbb{F}_q^\times\); \(GG[0]\) is the
trivial representation of the trivial group). It only remains to calculate the derivatives
\(\Sigma^{n-i}\) of \(\Sigma\) which follows from proposition 16.1 (the Leibnitz rule about derivatives).

As a simple consequence of this corollary, we have the following.

**Corollary 16.4.** Let \(n = n_1 + \cdots + n_r\) be a sum of positive integers, and let \(\pi_i\) be
an irreducible cuspidal representation of \(GL(n_i, \mathbb{F}_q)\). Assume that the representations
\(\pi_1, \cdots, \pi_r\) consist of distinct representations, so that \(\pi_1 \times \cdots \times \pi_r\), the corresponding
parabolically induced representation of \(GL(n, \mathbb{F}_q)\) is irreducible. Similarly, let \(n-1 = m_1 + \cdots + m_s\) be a sum of positive integers, and let \(\mu_i\) be an irreducible cuspidal
representation of \(GL(m_i, \mathbb{F}_q)\). Assume that the representations \(\mu_1, \cdots, \mu_s\) consist of
distinct representations, so that \(\mu_1 \times \cdots \times \mu_s\), the corresponding parabolically induced
representation of \(GL(n-1, \mathbb{F}_q)\) is irreducible. Then the restriction of \(\pi_1 \times \cdots \times \pi_r\) to
\(GL(n-1, \mathbb{F}_q)\) contains the representation \(\mu_1 \times \cdots \times \mu_r\) of \(GL(n-1, \mathbb{F}_q)\) with multiplicity
\(2^d\),

where \(d\) is the cardinality of the set of common representations in \(\{\pi_1, \cdots, \pi_r\}\) and
\(\{\mu_1, \cdots, \mu_s\}\).

17. **Branching laws for \(U(n, \mathbb{F}_q)\) via base change**

We use the method of base change, also called Shintani descent, to deduce some conclusions
about branching laws for the restriction of a representation of \(U(n, \mathbb{F}_q)\) to \(U(n-1, \mathbb{F}_q)\)
from the corresponding results for general linear groups obtained in the previous section. We
make crucial use of multiplicity 1 theorem for unitary groups which has recently been proved
for non-Archimedean fields by Aizenbud, Gourevitch, Rallis and Schiffmann in [AGRS]. We
begin with a brief review of what is called the Shintani descent.
Let $G$ be a connected reductive algebraic group over $\mathbb{F}_q$. Let $m \geq 1$ be a fixed integer. The group $G(\mathbb{F}_{q^m})$ comes equipped with its Frobenius automorphism $F$, whose fixed points are $G(\mathbb{F}_q)$. There is a natural map, called the norm mapping, from $F$-conjugacy classes in $G(\mathbb{F}_{q^m})$ to conjugacy classes in $G(\mathbb{F}_q)$ which is an isomorphism of the corresponding sets, and is furthermore an isometry:

$$\langle \chi_1, \chi_2 \rangle_{G(\mathbb{F}_q)} = \langle \chi'_1, \chi'_2 \rangle_{G(\mathbb{F}_{q^m})},$$

where $\chi_1$ and $\chi_2$ are class functions on $G(\mathbb{F}_q)$ which are related through the norm mapping to $F$-conjugacy class functions $\chi'_1$ and $\chi'_2$ on $G(\mathbb{F}_{q^m})$; the inner products are normalized so that

$$\langle 1, 1 \rangle_{G(\mathbb{F}_q)} = \langle 1, 1 \rangle_{G(\mathbb{F}_{q^m})} = 1.$$

We recall that according to Deligne-Lusztig, given a maximal torus $T$ of $G$ defined over $\mathbb{F}_q$, and a character $\theta : T(\mathbb{F}_q) \to \mathbb{C}^*$, there is a (virtual) representation of $G(\mathbb{F}_q)$ denoted by $R(T, \theta)$, now called the Deligne-Lusztig representation.

Given a character $\theta : T(\mathbb{F}_q) \to \mathbb{C}^*$, there are characters $\theta_m : T_m = T(\mathbb{F}_{q^m}) \to \mathbb{C}^*$ obtained by composing with the norm mapping: $T(\mathbb{F}_{q^m}) \to T(\mathbb{F}_q)$.

It is a basic fact that the characters of $R(T_m, \theta_m)$ and $R(T, \theta)$ are related by the norm mapping. We assume in what follows that $m = 2$, and that $R(T_2, \theta_2)$ is an irreducible representation of $G(\mathbb{F}_{q^2})$ which is invariant under the action of $\langle F \rangle = \text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$, and therefore extends to an irreducible representation of $G(\mathbb{F}_{q^2}) \rtimes \langle F \rangle$.

Since one can use irreducible cuspidal representation of $U(n, \mathbb{F}_q)$ to construct irreducible cuspidal representation of $U(n, k)$ where $k$ is the local field with maximal compact subring the Witt ring of $\mathbb{F}_q$, the following is an easy consequence of multiplicity 1 theorem for $p$-adic groups; we will not give a proof.

**Proposition 17.1.** Let $\pi_1$ be an irreducible cuspidal representation of $U(n, \mathbb{F}_q)$, and $\pi_2$ be an irreducible cuspidal representation of $U(n-1, \mathbb{F}_q)$. Then $\pi_2$ appears in $\pi_1$ restricted to $U(n-1, \mathbb{F}_q)$ with multiplicity at most 1.

With notation as in the proposition, let $\chi_1$ be the character of the representation $\pi_1$ and $\chi_2$ that of $\pi_2$. Let $\chi'_1$ and $\chi'_2$ be the characters of the corresponding representations of $GL(n, \mathbb{F}_{q^2})$ and $GL(n-1, \mathbb{F}_{q^2})$ obtained by base change, which we assume are irreducible, and hence extend to representations of semi-direct products: $GL(n, \mathbb{F}_{q^2}) \rtimes \langle F \rangle, GL(n-1, \mathbb{F}_{q^2}) \rtimes \langle F \rangle$. We denote the characters of these representations of $GL(n, \mathbb{F}_{q^2}) \rtimes \langle F \rangle$ and $GL(n-1, \mathbb{F}_{q^2}) \rtimes \langle F \rangle$ also by $\chi'_1$ and $\chi'_2$. It follows that

$$\langle \chi'_1, \chi'_2 \rangle_{GL(n-1, \mathbb{F}_{q^2}) \rtimes \langle F \rangle} = \langle \chi'_1, \chi'_2 \rangle_{GL(n-1, \mathbb{F}_{q^2})}.$$

Therefore,

$$\langle \chi'_1, \chi'_2 \rangle_{GL(n-1, \mathbb{F}_{q^2}) \rtimes \langle F \rangle} = \langle \chi'_1, \chi'_2 \rangle_{GL(n-1, \mathbb{F}_{q^2})} + \langle \chi_1, \chi_2 \rangle_{U(n-1, \mathbb{F}_q)}.$$

Now we observe that the left hand side of this last equality is an even integer (by Schur Orthogonality, observing that the group $GL(n-1, \mathbb{F}_{q^2}) \rtimes \langle F \rangle$ has been given volume 2),
whereas by our calculations of the last section,  \( \langle \chi_1', \chi_2' \rangle_{GL(n-1, \mathbb{F}_q)} \) is either an even integer if the cuspidal supports of \( \chi_1' \) and \( \chi_2' \) are not disjoint, or equals 1 if the cuspidal supports are disjoint. The term  \( \langle \chi_1, \chi_2 \rangle_{U(n-1, \mathbb{F}_q)} \) is either 0 or 1 by Proposition 17.1.

Therefore we get the following theorem as our only option.

**Theorem 17.2.** Let \( \pi_1 \) be an irreducible cuspidal representation of \( U(n, \mathbb{F}_q) \), and \( \pi_2 \) an irreducible cuspidal representation of \( U(n-1, \mathbb{F}_q) \). Assume that both \( \pi_1 \) and \( \pi_2 \) are Deligne-Lusztig representations. Then \( \pi_2 \) appears in the restriction of \( \pi_1 \) if and only if for the representations \( \pi_1' \) and \( \pi_2' \) which are the base change of \( \pi_1 \) and \( \pi_2 \) to \( GL(n, \mathbb{F}_q) \) and \( GL(n-1, \mathbb{F}_q) \), the cuspidal supports are disjoint.

**Remark :** A Deligne-Lusztig representation \( R(T, \theta) \) is known to be irreducible if and only if \( \theta \), a character of \( T(\mathbb{F}_q) \), is not stabilized by any non-trivial element of \( W(\mathbb{F}_q) \), where \( W \) is the Weyl group of \( T \). It can be seen that the base change of such a character of \( T(\mathbb{F}_q) \) to \( T(\mathbb{F}_q^n) \) continues to have this property, and as a result, the base change of an irreducible Deligne-Lusztig representation is an irreducible Deligne-Lusztig representation.

### 18. Depth zero supercuspidals

We now test the local conjecture in the \( p \)-adic case, for some tamely ramified discrete parameters, using the calculation of root numbers in \( \S 11 \). We will consider parameters

\[ \varphi : W_k \to \hat{G} \rtimes \text{Gal}(K/k) \]

trivial on \( \text{SL}_2(\mathbb{C}) \), with \( K/k \) unramified. We will further assume that \( \varphi \) is tamely ramified and that the centralizer of the image of inertia is the maximal torus \( \hat{T} \) of \( \hat{G} \). Finally we will assume that the image of the Frobenius in the quotient group \( N_L G(\hat{T})/\hat{T} \) acts as \(-1\) on \( X(\hat{T}) \). It would be interesting to test the local conjecture on more general tamely ramified parameters.

Under these conditions, \( \varphi \) determines a Langlands parameter \( \varphi(T) \) for the unramified anisotropic torus \( T = U(1)^n \times U(1)^{n-1} \) over \( k \). By local class field theory, we obtain a tame regular character \( \rho : T(k) \to \mathbb{C}^\times \), which factors through the quotient \( T(\mathbb{F}_q) \). If we identify \( T(\mathbb{F}_q) \) with the product \( (\mathbb{F}_q^\times)^n \times (\mathbb{F}_q^\times)^{n-1} \), then \( \rho \) is given by the collection \( (\alpha_1, \beta_2) \) as in \( \S 11 \). The centralizer of our parameter \( \varphi \) is given by

\[ C_\varphi = A_\varphi = \hat{T}[2] = \langle \pm 1 \rangle^n \times \langle \pm 1 \rangle^{n-1}. \]

We first show how a character \( \lambda : A_\varphi \to \langle \pm 1 \rangle \) corresponds to an embedding of torus \( T \) into a pure inner form \( G' \) of \( G \). We do this analysis for each unitary space \( W \) and \( W_0 \) separately.

An embedding of the torus \( S = U(1)^n \) into the unitary group \( U(W) \) corresponds to a decomposition of \( W \) into orthogonal lines over \( K \), each stable under the action of \( S \):
$W = \bigoplus_{i=1}^n Ke_i$. The conjugacy class of the embedding $f : S \to U(W)$ depends on the discriminants

$$\langle e_i, e_i \rangle \text{ in } k^X/\mathbb{N}K^X.$$ 

Since $K/k$ is unramified, it depends on the signs

$$(−1)^{\text{ord}(e_i, e_i)} \text{ in } \langle ±1 \rangle^n.$$ 

There are $2^{n−1}$ possible conjugacy classes, as we have the relation

$$\prod \langle e_i, e_i \rangle \equiv \text{disc}(W) \mod NK^X.$$ 

Since the 2 pure inner forms $W$ and $W'$ have distinct discriminants, there are $2^n$ embeddings of $S$ into either $U(W)$ or $U(W')$.

Given an embedding $f : S \to U(W')$, there is a unique maximal compact subgroup $K_f \subset U(W')$ which contains the image of $S$. This is the subgroup stabilizing the lattice, $L_f = \bigoplus_{i=1}^n AKe_i$, where the orthogonal vectors $e_i$ are chosen on the $S$-stable lines to satisfy $\text{ord}(e_i, e_i) = 0, 1$. The reduction of $\bar{K}_f(\mod \pi)$ has reductive quotient

$$\bar{K}_f \cong U(p) \times U(n−p),$$

over $\mathbb{F}_q$, where $p$ is the number of $e_i$ with $(-1)^{\text{ord}(e_i, e_i)} = −1$. Hence $\bar{K}_f$ will be hyperspecial if and only if all of the inner products $\langle e_i, e_i \rangle$ have valuations of the same parity. This justifies the claim made in §10, before the statement of Conjecture 10.4.

From the calculation of the character $\chi$ of $A_3$ in Corollary 11.2, we conclude that the associated embedding

$$f : U(W_0') \times U(W')$$

has image contained in the maximal compact subgroup $K_f$ of $G'$ with reduction isomorphic to

$$\bar{G}' = \bar{K}_f = (U(p) \times U(n−1−p)) \times (U(p) \times U(n−p))$$

over $\mathbb{F}_q$. Here $p$ is the number of pairs $(\alpha^0_i, \beta_i)$ with $\alpha^0_i \beta_i = 1$.

The regular character $\rho$ of $T(\mathbb{F}_q)$ allows us to construct an irreducible, supercuspidal representation $R(T, \rho)$ of the finite group $\bar{K}_f(\mathbb{F}_q)$, using the method of Deligne and Lusztig. Since $\bar{K}_f(\mathbb{F}_q)$ is a direct product, we may write $R(T, \rho)$ as the tensor product of 4 terms

$$(R_1 \otimes R_2) \otimes (R_3 \otimes R_4).$$

Since $\alpha^0_i \beta_i = 1$ for $1 ≤ i ≤ p$, we conclude that $R_1$ and $R_3$ are dual irreducible representations of the finite group $U(p)$. In particular, there is a $U(p)$-invariant linear form $R_1 \otimes R_3 \to \mathbb{C}$, unique up to scaling.

To prove the existence of (unique) $\bar{H}' = U(p) \times U(n−1−p)$-invariant linear form on $R(T, \rho)$, we must establish the existence of a unique $U(n−1−p)$-invariant linear form $R_2 \otimes R_4 \to \mathbb{C}$. In other words, we must show that the dual of $R_2$ occurs with multiplicity 1 in the restriction of $R_4$ from $U(n−p)$ to $U(n−p−1)$. This follows from Theorem 17.2, as the tame characters $\alpha^0_i$ and $\beta_i$ giving $R_2$ and $R_4$ satisfy $\alpha^0_i \beta _i ≠ 1$ for all $i, j$. 
Since $\pi' = \text{Ind}_{G'}^G(R(T, \rho))$, the $H'$-invariant linear form on $R(T, \rho)$ gives a non-zero $H'$-invariant form on $\pi'$. Thus $\text{Hom}_{H'}(\pi', C)$ is non-zero, as predicted.

19. Global implies local: some partial results, I

The aim of this section is to prove the following local theorem by global methods.

**Theorem 19.1.** Let $W_0$ be a 2 dimensional non-degenerate Hermitian subspace of a Hermitian space $W$ of dimension 3 over a non-Archimedean local field $K$, with $K$ quadratic over $k$. Suppose that $\pi_0$ (resp. $\pi_1$) is an irreducible representation of $U(W_0)$ (resp. $U(W_1)$) which belongs to a generic $L$-packet. Let the Langlands parameter of $\pi_0$ (resp. $\pi_1$) restricted to $K$ be $\sigma_0$ (resp. $\sigma_1$). Suppose that $\text{Hom}_{U(W_0)}(\pi_0 \otimes \pi_1, C) \neq 0$.

Then

$$\epsilon(\sigma_0 \otimes \sigma_1) = \begin{cases} 1 & \text{if } U(W_0) \times U(W_1) \text{ is quasi-split} \\ -1 & \text{otherwise} \end{cases}$$

**Remark:** The method that we follow to prove this theorem is pretty general, but it is based on a global theorem of Ginzburg, Jiang, and Rallis [GJR, theorem 4.6] which assumes that automorphic forms on unitary groups $U(n)$ have base change to $GL(n)$ something which is known at the moment only for generic automorphic representations on quasi-split unitary groups. However, by Rogawski [Ro], base change is known for any unitary group in 3 variables, which is why we have restricted ourselves to $U(3)$ in the above theorem. Nonetheless, we have formulated some of the preliminary results below in greater generality.

We begin with the following globalization result about local fields, which will be applied to globalize Hermitian spaces over local fields so that there is no ramification outside the place being considered, and the unitary groups at infinity are either compact, or of rank 1.

**Lemma 19.2.** Let $K$ be a quadratic extension of a non-Archimedean local field $k$. Then there exists a totally real number field $F$ with $k$ as its completion, and a quadratic totally imaginary extension $E$ of $F$ with corresponding completion $K$ such that $E$ is unramified over $F$ at all finite places different from $K$.

**Proof:** Except for the requirement about $E$ being unramified except for the place $K$, this is well-known. Suppose that a quadratic extension $E_1$ over $F_1$ with possible ramifications is constructed. Then a well-known technique, crossing with a field, says that after a suitable base change, one can get rid of the ramifications; we leave the details to the reader.

**Lemma 19.3.** Let $W$ be a non-degenerate Hermitian space over a non-Archimedean local field $K$, with $K$ quadratic over $k$. Let $F$ be a totally real number field with completion $k$ at a place of $F$, and let $E$ be a quadratic totally imaginary extension of $F$ with corresponding completion $K$. Then there is a Hermitian space $V$ over $E$ giving rise to $W$ over $K$ in such a way that the corresponding unitary group is quasi-split at all finite places of $F$ except the one corresponding to the completion $k$; and at all but one infinite place the group is the compact group $U(n)$, and at the remaining infinite
place, the group is either $U(n)$, or $U(n-1,1)$; if $n$ is odd, we can assume that the group is compact at all the infinite places.

**Proof:** The proof of the lemma will depend on the well-known classification of a Hermitian form over a number field, according to which a Hermitian form over a number field is determined by

1. the normalized discriminant, and
2. the signatures at the infinite places.

Moreover, given any normalized discriminant, and signatures at infinite places (except for obvious compatibility between normalized discriminant and signatures), there is a Hermitian form.

We also note the following exact sequence from classfield theory,

$$0 \to F^\times/NE^\times \to \mathbb{A}_F^\times/NA_E^\times \to \text{Gal}(E/F) \to 0,$$

from which it follows that one can construct an element in $F^\times$ which is trivial in $F_v^\times/NE_v^\times$ at all the finite places except $v$, and which at the infinite places has the desired signs, except that the product of the signs is 1 or -1, depending on whether the element in $k^\times/Nk^\times$ is trivial or non-trivial.

The proof of the lemma is now completed by observing that a Hermitian form of normalized discriminant 1 over a non-Archimedean local field defines a quasi-split group, and that the normalized discriminants of the Hermitian spaces $Z_1Z_2 + \cdots + Z_nZ_n$, and $Z_1Z_1 + Z_2Z_2 + \cdots + Z_nZ_n$ over $\mathbb{C}$ are negative of each other, and if $n$ is odd, the normalized discriminants of the Hermitian spaces $Z_1Z_1 + Z_2Z_2 + \cdots + Z_nZ_n$, and $-(Z_1Z_2 + \cdots + Z_nZ_n)$ are negative of each other.

**Corollary 19.4.** Let $W_0$ be a non-degenerate Hermitian subspace of codimension 1 of a Hermitian space $W_1$ over a non-Archimedean local field $K$, with $K$ quadratic over $k$. Let $F$ be a totally real number field with completion $k$ at a place of $F$, and let $E$ be a quadratic totally imaginary extension of $F$ with corresponding completion $K$. Then there is a Hermitian subspace $V_0$ of codimension 1 of a Hermitian space $V_1$ over $E$ giving rise to $W_0$ and $W_1$ over $K$ in such a way that the corresponding unitary groups are quasi-split at all the finite places of $F$ except the one corresponding to the completion $k$; assuming $F \neq \mathbb{Q}$, the group $U(V_1)$ is the compact group $U(n+1)$ at all but two infinite places, and at the remaining infinite places, the group is either $U(n+1)$, or $U(n,1)$; the subgroup $U(V_0)$ is compact at all but one infinite place.

**Proof:** Let $W_1 = W_0 \oplus L_c$ where $L_c$ is $K$ with the Hermitian structure $cZ\bar{Z}$, $c \in k^\times$. Globalize $W_0$ by the previous lemma, and globalize $c$ so that the normalized discriminant of $W_1$ is 1 at all the finite places of $F$ other than $k$ (so that $U(V_1)$ is quasi-split at all the finite places of $F$ other than $k$), and so that $c$ has arbitrary signs at infinity, with only the product of the signs pre-determined which allows for the desired conclusion.

We omit a proof of the following corollary of the lemma which follows exactly as in the previous corollary.
Corollary 19.5. If \( n \equiv 1, 2 \mod 4 \), then Hermitian spaces \( W_0 \subset W_1 \) of dimensions \( n, n+1 \) can be globalized keeping them positive definite at infinity, and maximally split at all finite places other than \( K \). For \( n \equiv 2 \mod 4 \), if \( W_0 \) has an isotropic subspace of dimension \( n/2 \), and for \( n \equiv 1 \mod 4 \), if \( W_1 \) has an isotropic subspace of dimension \( (n+1)/2 \), then there are an even number of real places in \( F \), else an odd number of real places.

Proof of Theorem 19.1: If the representation \( \pi_0 \) is an irreducible principal series representation of \( U(W_0) \), or is contained in a unitary principal series representation, then
\[
\sigma_0 = \alpha \oplus \bar{\alpha}^\vee.
\]
It follows that
\[
\sigma_0 \otimes \sigma_1 = \tau \oplus \bar{\tau}^\vee,
\]
for \( \tau = \alpha \otimes \sigma_1 \), and therefore by lemma 9.3,
\[
\epsilon(\sigma_0 \otimes \sigma_1) = 1.
\]
As the principal series representation occurs only on the quasi-split form of \( U(2) \), this proves the conclusion desired in the theorem.

If the representation \( \pi_1 \) is an irreducible principal series, then the theorem is a consequence of our analysis in section 15.

It is easy to see that the Steinberg representation of \( U(1, 1) \) is a quotient of the Steinberg representation of \( U(2, 1) \) (by simply restricting functions on the flag variety of \( U(2, 1) \) to the flag variety of \( U(1, 1) \)), and that (denoting \( st_n \) the \( n \)-dimensional irreducible representation of the \( SL_2(\mathbb{C}) \) part of \( W'_K \))
\[
\epsilon(st_3 \otimes st_2) = \epsilon(st_4 \oplus st_2) = 1.
\]
It therefore suffices to assume for the rest of the proof that both the representations \( \pi_0 \) and \( \pi_1 \) are supercuspidal.

Since the group \( U(V_1) \) is compact at infinity, it is easy to see that we can globalize the representation \( \pi_1 \) of \( U(W_1) \) to an automorphic representation \( \Pi_1 \) in \( U(V_1)(A) \) in such a way that it is unramified at all the finite places of \( F \) except \( k \).

By Lemma 1 of [P3], we can globalize \( \pi_0 \) to an automorphic representation \( \Pi_0 \) such that the period integral
\[
\int_{U(V_0)\backslash U(V_0)(A)} f_0 f_1 \neq 0,
\]
for some \( f_0 \) in \( \Pi_0 \), and \( f_1 \) in \( \Pi_1 \).

By the theorems due to Ginzburg, Jiang, and Rallis, cf. [JGR2, theorem 4.6], since the period integral is nonzero, the central critical \( L \)-value,
\[
L\left( \frac{1}{2}, \Pi_0^E \otimes \Pi_1^E \right) \neq 0,
\]
where \( \Pi_0^E \) and \( \Pi_1^E \) denote base change of \( \Pi_0 \) and \( \Pi_1 \) to \( E \).
This implies that the global root number,
\[ \epsilon\left(\frac{1}{2}, \Pi_0^E \otimes \Pi_1^E\right) = 1. \]

Let
\[ \Pi_0 = \otimes_w \Pi_{0,w}, \text{ and } \Pi_1 = \otimes_w \Pi_{1,w}, \]
with \( \Pi_{0,w} = \pi_0 \) and \( \Pi_{1,w} = \pi_1 \). From the nonvanishing of the period integral, it follows that
\[ \text{Hom}_{U(1,W)}(\Pi_{0,w} \otimes \Pi_{1,w}, \mathbb{C}) \neq 0. \]

Since the representations \( \Pi_{1,w} \) for \( w \), a finite place of \( F \), not \( v \), are unramified by construction, they are in particular quotients of principal series representations. It follows from section 15, or also by a direct calculation involving Mackey theory, that
\[ \epsilon_w\left(\frac{1}{2}, \Pi_0^E \otimes \Pi_1^E\right) = 1. \]

Since the global epsilon factor is a product of local epsilon factors, and since the representation \( \Pi_1 \) is unramified at all finite places of \( F \) except that corresponding to \( k \), we have
\[ \epsilon\left(\frac{1}{2}, \sigma_0 \otimes \sigma_1\right) = \{ 1 \text{ if } n \equiv 0, 3 \mod 4, -1 \text{ if } n \equiv 1, 2 \mod 4. \]

The following lemmas then complete the proof of the theorem on noting that there are an even number of places at infinity if \( U(W_0) \) is quasi-split, and odd number of places at infinity when \( U(W_0) \) is not quasi-split.

**Lemma 19.6.** Let \( W_0 \) be a codimension 1 Hermitian subspace of a positive definite Hermitian space \( W \) of dimension \( n + 1 \) over \( \mathbb{C} \). Suppose that \( \pi_0 \) (resp. \( \pi_1 \)) is a finite dimensional irreducible representation of \( U(W) \) (resp. \( U(W_0) \)). Let the Langlands parameter of \( \pi_0 \) (resp. \( \pi_1 \)) restricted to \( K \) be \( \sigma_0 \) (resp. \( \sigma_1 \)). Suppose that \( \text{Hom}_{U(W_0)}(\pi_0 \otimes \pi_1, \mathbb{C}) \neq 0. \) Then
\[ \epsilon(\sigma_0 \otimes \sigma_1) = \begin{cases} 1 & \text{if } n \equiv 0, 3 \mod 4, \\ -1 & \text{if } n \equiv 1, 2 \mod 4. \end{cases} \]

**Proof:** The proof of this lemma is a simple consequence of the well-known branching law from the compact group \( U(n+1) \) to \( U(n) \), combined with the value of the epsilon factor given by the following lemma.

**Lemma 19.7.** Let \( \psi \) be the additive character on \( \mathbb{C} \) given by \( \psi(z) = e^{-2\pi iy} \) where \( z = x+iy \). For \( n \) an integer, let \( \chi_n \) denote the character \( \chi_n(z) = e^{ni\theta} \) for \( z = re^{i\theta} \in \mathbb{C}^\times \). Then for \( n \) odd,
\[ \epsilon(\chi_n, \psi) = \begin{cases} 1 & \text{if } n > 0, \\ -1 & \text{if } n < 0. \end{cases} \]
Lemma 19.8. Let $\pi_0$ (resp. $\pi_1$) be a finite dimensional irreducible representation of the compact group $U(n)$ (resp. $U(n+1)$) with $L$-parameter restricted to $\mathbb{C}^\times$ given by an $n$-tuple of half-integers $\sigma_0 = \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ (resp. \{\mu_1 < \mu_2 < \cdots \mu_{n+1}\}) an $(n+1)$-tuple of half-integers), where all the $\lambda_i$’s are half-integers but not integers if $n$ is even, and are integers in $n$ is odd, and $\mu_i$’s are all integers if $n$ is even, and half-integers but not integers if $n$ is odd. Then $\pi_0$ appears in $\pi_1$ restricted to $U(n)$ if and only if $$\mu_1 < \lambda_1 < \mu_2 < \cdots < \lambda_n < \mu_{n+1}.$$ 

Corollary 19.9. With notation as in the lemma, and assuming that $\pi_0$ appears in $\pi_1$ $$\epsilon(\mu_k \otimes \sigma_0) = (-1)^{k-1}, \text{ for all } k,$$ and therefore, $$\epsilon(\sigma_1 \otimes \sigma_0) = \prod_{k=1}^{n+1} (-1)^{k-1} = (-1)^{\frac{n(n+1)}{2}}.$$ 

Remark: Notice that $\epsilon(\mu_k \otimes \sigma_0) = (-1)^{k-1}, \text{ for all } k,$ is independent of the representation $\pi_0$ of $U(n)$ as long as it appears in the representation $\pi_1$ of $U(n+1)$, exactly as one of the formulations of our local conjecture in §6.

Part 3. BESSEL and FOURIER-JACOBI MODELS

In Part 3 of this paper, we shall consider certain generalizations of the restriction problem studied in Part 1 and formulated certain extensions of Conjectures 6.1, 6.2. These extensions are analogs of those treated in [GP2] for the case of orthogonal groups. So let us briefly revisit the case of orthogonal groups here.

We will deal exclusively with local fields in this part. There are natural global periods too in this context, for which the answer will depend on

1. Existence of local Bessel or Fourier-Jacobi models.
2. Non-vanishing of a certain $L$-value at the center of the critical strip.

As these considerations are totally analogous to what we have discussed in [GP1] and in this paper for $U(n), U(n-1)$, we will not dwell on it at all.

In [GP1], the restriction problem for the pair $(SO(2n+1), SO(2n))$ or $(SO(2n), SO(2n-1))$ was studied and the proposed solution is expressed in terms of root numbers constructed out of the symplectic representation $$W_k' \rightarrow L(SO(2n+1) \times SO(2n)) \subset Sp_{2n}(\mathbb{C}) \times O_{2n}(\mathbb{C}),$$
or

\[ W'_k \longrightarrow L^1(SO(2n-1) \times SO(2n)) \subset \text{Sp}_{2n-2}(\mathbb{C}) \times \text{O}_{2n}(\mathbb{C}). \]

From the Galois theoretic point of view, it is natural to consider general symplectic representations of \( W'_k \) of the form

\[ W'_k \longrightarrow L^1(SO(2m+1) \times SO(2n)) \subset \text{Sp}(2m) \times \text{O}_{2n}(\mathbb{C}), \]

i.e. obtained by the tensor product of a symplectic representation and an orthogonal representation of even dimension. This should correspond to a restriction problem for the pair \((SO(2m+1), SO(2n))\) corresponding to a pair

\[ W \subset V \]

of quadratic spaces of odd codimension. As explained in [GP2], such a restriction problem can be formulated in terms of Bessel models. One of our main results in Part 3 is that the conjecture for Bessel models as formulated in [GP2] is a consequence of that for \((SO(n), SO(n-1))\) as formulated in [GP1].

On the other hand, one can go one step further and consider symplectic representations of the form

\[ W'_k \longrightarrow \text{Sp}_{2m}(\mathbb{C}) \times \text{SO}_{2n+1}(\mathbb{C}), \]

i.e. obtained by the tensor product of a symplectic representation and an orthogonal representation of odd dimension. What restriction problem might this correspond to? As we shall see in Sections 22 and 24, the corresponding restriction problem is one for the metaplectic-symplectic pair

\[ \text{Sp}(2m) \times \text{Sp}(2n) \]

where \( \text{Sp}(2m) \) denotes the unique two-fold cover of \( \text{Sp}(2m) \). This restriction problem is formulated in terms of Fourier-Jacobi models. In particular, we are following the folklore that the \( L \)-group of \( \text{Sp}(2m) \) is \( \text{Sp}_{2n}(\mathbb{C}) \). As we shall see in Section 23, this is confirmed by a recent result of Kudla-Rallis [KR] which classifies the irreducible representations of \( \text{Sp}(2m) \) in terms of those of \( \text{SO}(2n+1) \).

Returning to unitary groups, we recall that the problem studied in Parts 1 and 2 is the restriction of an irreducible representation of \( U(n) \) to \( U(n-1) \) and the proposed solution involves the root number associated to a natural \( 2n(n-1) \)-dimensional representation of

\[ L(U(n) \times U(n-1)) = (\text{GL}_n(\mathbb{C}) \times \text{GL}_{n-1}(\mathbb{C})) \rtimes \text{Gal}(K/k). \]

As above, it is natural to consider restriction problems for a pair \((U(W), U(V)) = (U(m), U(n))\) where

\[ W \subset V \]

is a pair of Hermitian spaces of odd codimension, so that the \( L \)-group

\[ L(U(m) \times U(n)) = (\text{GL}_m(\mathbb{C}) \times \text{GL}_n(\mathbb{C})) \rtimes \text{Gal}(K/k) \]
has a natural $2mn$-dimensional symplectic representation. As we shall see in Section 20, the corresponding restriction problem is expressed in terms of the *Bessel models*. However, as in the orthogonal case, our conjecture for Bessel models for unitary groups turns out to be a consequence of the conjectures formulated in Part 1 of this paper.

On the other hand, if one considers $(U(m), U(n))$ where $m$ and $n$ have the same parity, then the corresponding restriction problem is expressed in terms of *Fourier-Jacobi models*. In this case, the $L$-group of $U(m) \times U(n)$ actually has a natural orthogonal representation, but one which can be twisted slightly to create a symplectic representation as we will see in Section 25.

20. **Bessel Models for Unitary Groups**

As in [GP2], one can formulate a restriction problem from $U(V)$ to a smaller $U(W)$ where $W \subset V$ has odd codimension $> 1$. This proceeds by the consideration of Bessel models. The purpose of this section is to introduce these Bessel models and to formulate a conjecture on their non-vanishing.

Let $W$ be a Hermitian subspace of a Hermitian space $V$ of odd codimension such that $W^\perp$ is a split Hermitian space. Hence, we may write

$$ V = X \oplus W \oplus \langle e \rangle \oplus X^\vee $$

where $X$ and $X^\vee$ are isotropic subspaces of $V$ in duality with each other and $e$ is a non-isotropic vector in $W^\perp$. Let $P(X)$ be the parabolic subgroup in $U(V)$ stabilizing the subspace $X$, and let $M(X)$ be the Levi subgroup of $P(X)$ which stabilizes both $X$ and $X^\vee$. Then

$$ M(X) \cong \text{GL}_K(X) \times U(W \oplus \langle e \rangle) $$

stabilizes $Y = X \oplus X^\vee$ and $W \oplus \langle e \rangle$ as well. We have

$$ P(X) = M(X) \rtimes N(X) $$

where $N(X)$ is the unipotent radical of $P(X)$ and sits in an exact sequence of $M(X)$-modules,

$$ 0 \to \tilde{\Lambda}^2X \to N(X) \to X \otimes (W \oplus \langle e \rangle) \to 0 $$

with $\tilde{\Lambda}^2(X)$ denoting the space of skew-Hermitian forms on $X^\vee$.

Let $\ell_1 : X \to \mathbb{G}_a$ be a nonzero homomorphism, and let

$$ \ell_W : W \oplus \langle e \rangle \to \mathbb{G}_a $$

be a nonzero homomorphism which is zero on the hyperplane $W$. This gives a map

$$ \ell_1 \otimes \ell_W : X \otimes (W \oplus \langle e \rangle) \to \mathbb{G}_a, $$

and one can consider the composite map,

$$ m : N(X) \to N(X)^{ab} = X \otimes (W \oplus \langle e \rangle) \to \mathbb{G}_a. $$
The subgroup of $M(X)$ which stabilizes the map $m$ is

$$\text{GL}(X)_{\ell_1} \times U(W)$$

where $\text{GL}(X)_{\ell_1}$ is the stabilizer in $\text{GL}(X)$ of the linear form $\ell_1$. Let $U_X$ be the maximal unipotent subgroup in this mirabolic subgroup $\text{GL}(X)_{\ell_1}$ of $\text{GL}(X)$. Define the subgroup $N$ of $P(X)$ by

$$N = U_X \ltimes N(X).$$

Let $\ell_X : U_X \to \mathbb{G}_a$ be a homomorphism which is nontrivial on each simple root space of $\text{GL}(X)$ in $U_X$, so that $\ell_X$ is a generic character. There is then a unique homomorphism

$$\ell : N \to \mathbb{G}_a$$

which is equal to $\ell_X$ on the subgroup $U_X$, and equal to $m$ on the subgroup $N(X)$. If $\psi$ is a non-trivial additive character on $k$, then we obtain by composition a character

$$\psi_\ell : N(k) \to k \to \mathbb{C}^\times$$

of $N$.

Now we have the following proposition.

**Proposition 20.1.** The pair $(N, \ell)$ is uniquely determined up to conjugacy in the group $U(V)$ by the pair $W \subset V$ (with $W^\perp$ split). In particular, the pair $(N, \psi_\ell)$ is also determined up to conjugacy in $U(V)$ by $W \subset V$.

We define the **Bessel subgroup** $B_{W,X}$ to be

$$B_{W,X} := \text{SO}(W) \ltimes N = (\text{SO}(W) \times U_X) \ltimes N(X).$$

By the proposition, $B_{W,X}$ depends only on $W \subset V$ up to conjugacy. Observe that $U(W)$ fixes the character $\psi_\ell$ and so we may extend $\psi_\ell$ to $B_{W,X}$ so that it is trivial on $U(W)$. The pair $(B_{W,X}, \psi_\ell)$ only depends on $W \subset V$ up to conjugacy. As a result, we may suppress the mention of $X$, $\ell$ or $\psi$ from further notation.

With the notation as above, we now come to the notion of a Bessel model.

**Definition (Bessel model):** Suppose that $W \subset V$ are Hermitian spaces such that $W^\perp$ is a split Hermitian space of odd dimension $2d + 1$ ($d \geq 0$). Let $\pi$ and $\pi_0$ be irreducible admissible representations of $U(V)$ and $U(W)$ respectively. Then the $\pi_0$-Bessel model of $\pi$ is the space

$$B_d(\pi, \pi_0) := \text{Hom}_{B_{W,X}}(\pi \otimes (\pi_0 \otimes \psi_\ell), \mathbb{C}).$$

We say that $\pi$ has $\pi_0$-Bessel model if the above Hom space is nonzero.

Note that when $d = 0$, then $W$ has codimension 1 in $V$ and $(N, \ell)$ is trivial, so that

$$B_0(\pi, \pi_0) = \text{Hom}_{U(W)}(\pi \otimes \pi_0, \mathbb{C})$$
is the space studied in Part 1 of this paper. Thus, the following conjecture should be regarded as an extension of Conjectures 6.1 and 6.2.

**Conjecture 20.2.**

1. For any irreducible representations \( \pi \) and \( \pi_0 \) of \( U(V) \) and \( U(W) \),
   \[
   \dim B_d(\pi, \pi_0) \leq 1.
   \]

2. Fix a generic Langlands parameter \( \varphi \) for \( U(V) \times U(W) \) with associated \( L \)-packet \( \Pi(\varphi) \). Then
   \[
   \sum_{\pi \otimes \pi_0 \in \Pi(\varphi)} \dim B_d(\pi, \pi_0) = 1,
   \]
   where the space \( B_d(\pi, \pi_0) \) is interpreted to be zero if \( \pi \otimes \pi_0 \) is a representation of an irrelevant \( U(V') \times U(W') \).

3. The unique representation \( \pi \otimes \pi_0 \) which has nonzero contribution to the sum in (2) is given by a character on the component group \( A_\varphi \), specified by the same recipe as in Conjecture 6.1.

When \( k \) is non-Archimedean, we shall see in Corollary 21.3 that the multiplicity one assertion in (1) follows from the multiplicity one theorem of [AGRS]. Moreover, we shall show that (2) follows from Conjecture 6.1.

### 21. Compatibility of Various Conjectures

The aim of this section is to prove that the Bessel model conjecture formulated in the previous section for general codimension \( 2d + 1 \) follows from the case when \( d = 0 \), i.e. is a consequence of Conjectures 6.1 and ???. Indeed, the same implication holds in the case of orthogonal groups, so that the expectations of [GP2] follow from those of [GP1]. Since it is easier to work with orthogonal groups than unitary groups, we shall give the details for the orthogonal case; the proof works essentially verbatim for the unitary case.

We begin by fixing some notation. Let

\[
V = X_m \oplus Y \oplus X_m^\vee
\]

be an \( n \)-dimensional quadratic space over a non-Archimedean local field, with

\[
X_m = \langle e_1, \ldots, e_m \rangle \quad \text{and} \quad X_m^\vee = \langle f_1, \ldots, f_m \rangle
\]

isotropic subspaces such that \( \langle e_i, f_j \rangle = \delta_{ij} \), and \( Y \) a non-degenerate quadratic space which is orthogonal to \( X_m \oplus X_m^\vee \). Let

\[
e_{mm} = e_m - f_m \quad \text{and} \quad f_{mm} = e_m + f_m,
\]

so that with \( X_{m-1} = \langle e_1, \ldots, e_{m-1} \rangle \), we have

\[
V = X_m \oplus Y \oplus X_m^\vee = X_{m-1} \oplus Y \oplus X_{m-1}^\vee \oplus e_{mm} \oplus f_{mm}.
\]
Let $P = P(X_m)$ be the parabolic subgroup of $SO(V)$ stabilizing the subspace $X_m$, and $M$ the Levi subgroup of $P$ stabilizing both $X_m$ and $X_m^\vee$, so that

$$M \cong GL(X_m) \times SO(Y).$$

Let $\tau$ be a supercuspidal representation of $GL(X_m)$, and $\pi_0$ an irreducible admissible representation of $SO(Y)$. Let

$$I(\tau, \pi_0) = Ind_P^SO(V) \tau \boxtimes \pi_0$$

be the (unnormalized) induced representation of $SO(V)$ from the representation $\tau \otimes \pi_0$ of $P$.

The aim of this section is to prove the following theorem, which computes the Hom space

$$B_0(I(\tau, \pi_0), \pi) = Hom_{SO(W)}(I(\tau, \pi_0), \pi^\vee).$$

**Theorem 21.1.** Let

$$W = X_{m-1} \oplus Y \oplus X_m^\vee \oplus e_{mm}$$

be a non-degenerate quadratic subspace of

$$V = X_m \oplus Y \oplus X_m^\vee$$

of codimension 1. Suppose that $W$ has an $m$-dimensional isotropic subspace $X_m' \supset X_{m-1}$ such that

$$W \cong X_m' \oplus Y' \oplus X_m'^\vee,$$

with $Y'$ a codimension 1 subspace of $Y$. Let $P(X_m')$ be the parabolic subgroup of $SO(W)$ stabilizing $X_m'$ with Levi subgroup $GL(X_m') \times SO(Y')$.

Let $\pi$ be an irreducible admissible representation of $SO(W)$. Assume that $\pi^\vee$ does not belong to the Bernstein component of $SO(W)$ associated to $(GL(X_m') \times SO(Y'), \tau \otimes \mu)$ for any irreducible representation $\mu$ of $SO(Y')$. Then we have:

$$B_0(I(\tau, \pi_0), \pi) \cong B_{m-1}(\pi, \pi_0).$$

In other words,

$$Hom_{SO(W)}(I(\tau, \pi_0), \pi^\vee) \cong Hom_{B_{Y',X_{m-1}}}(\pi, \pi_0^\vee \otimes \bar{\psi}_\ell).$$

where $(B_{Y,X_{m-1}}, \psi_\ell)$ is as defined in the previous section.

**Proof.** We calculate the restriction of $\Pi := I(\tau, \pi_0)$ to $SO(W)$ by Mackey’s orbit method. For this, we begin by observing that $SO(W)$ has two orbits on the flag variety $SO(V)/P(X_m)$ consisting of:

1. $m$-dimensional isotropic subspaces of $V$ which are contained in $W$; a representative of this orbit is the space $X_m'$ and its stabilizer in $SO(W)$ is the parabolic subgroup

$$P_W(X_m') = P(X_m') \cap SO(W);$$
(2) $m$-dimensional isotropic subspaces of $V$ which are not contained in $W$; a representative of this orbit is the space $X_m$ and its stabilizer in $\text{SO}(W)$ is the subgroup $H = P(X_m) \cap \text{SO}(W)$.

By Mackey theory, this gives a filtration on the restriction of $\Pi$ to $\text{SO}(W)$ as follows:

$$
0 \longrightarrow \text{ind}^{\text{SO}(W)}_H (\tau \otimes \pi_0)|_H \longrightarrow \Pi|_{\text{SO}(W)} \longrightarrow \text{Ind}^{\text{SO}(W)}_{\text{P}(X_m)} \tau \otimes \pi_0|_{\text{SO}(W')} \longrightarrow 0,
$$

where the induction functors here are unnormalized.

By our assumption, $\pi^\vee$ does not appear as a quotient of the 3rd term of the above short exact sequence and we have

$$
\text{Hom}_{\text{SO}(W)}(\text{ind}^{\text{SO}(W)}_H (\tau \otimes \pi_0)|_H, \pi^\vee) = \text{Hom}_{\text{SO}(W)}(\Pi, \pi^\vee).
$$

It thus suffices to analyze the representations of $\text{SO}(W)$ which appear on the open orbit. For this, we need to determine the group $H = P(X_m) \cap \text{SO}(W)$ as a subgroup of $\text{SO}(W)$ and $P(X_m)$.

Recall that $V = X_m \oplus Y \oplus X_m^\vee$, and $W$ is the codimension 1 subspace $X_{m-1} \oplus Y \oplus X_{m-1}^\vee \oplus e_{mm}$ which is the orthogonal complement of $f_{mm} = e_m + f_m$. It is not difficult to see that as a subgroup of $\text{SO}(W)$,

$$
H = \text{SO}(W) \cap P(X_m) \subset P_W(X_{m-1}).
$$

Indeed, if $g \in H$, then $g$ fixes $e_m + f_m$ and stabilizes $X_m$, and we need to show it stabilizes $X_{m-1}$. If $e \in X_{m-1}$, it suffices to show that $\langle g \cdot e, f_m \rangle = 0$. But

$$
\langle g \cdot e, f_m \rangle = \langle e, g^{-1} \cdot f_m \rangle = \langle e, f_m + e_m - g^{-1} \cdot e_m \rangle = 0,
$$

as desired. Now we claim that

$$
H = (\text{GL}(X_{m-1}) \times \text{SO}(Y)) \ltimes N_W(X_{m-1}) \subset P_W(X_{m-1}).
$$

To see this, given an element $h \in P_W(X_{m-1})$, we need to show that $h \cdot e_m \in X_m$ if and only if $h$ belongs to the RHS above. We know that $h \cdot (e_m + f_m) = e_m + f_m$ and we may write

$$
h \cdot (e_m - f_m) = \lambda \cdot (e_m - f_m) + y + e, \quad \text{with} \quad y \in Y \quad \text{and} \quad e \in X_{m-1}.
$$

Hence, $h \cdot e_m \in X_m$ if and only if $\lambda = 1$ and $y = 0$. Thus, $h$ fixes $e_m - f_m$ modulo $X_{m-1}$ and so stabilizes $Y$ modulo $X_{m-1}$. This implies that $h$ lies in $(\text{GL}(X_{m-1}) \times \text{SO}(Y)) \ltimes N_W(X_{m-1})$, so that we have a description of $H$ as a subgroup of $\text{SO}(W)$.

Since we are restricting the representation $\tau \otimes \pi_0$ of $P(X_m)$ to the subgroup $H$, we also need to know how $H$ sits in $P(X_m)$. For this, we have:
\[
0 \longrightarrow N(X_m) \longrightarrow P(X_m) \longrightarrow \text{GL}(X_m) \times \text{SO}(Y) \longrightarrow 0
\]
\[
0 \longrightarrow N(X_m) \cap H \longrightarrow H \longrightarrow E_{m-1} \times \text{SO}(Y) \longrightarrow 0
\]
where
\[
E_{m-1} \subset \text{GL}(X_m)
\]
is the mirabolic subgroup which stabilizes the subspace \(X_{m-1} \subset X_m\) and fixes \(e_m\). Note also that \(N(X_m) \cap H \subset N_W(X_{m-1})\) and
\[
N_W(X_m)/(N_W(X_m) \cap H) \cong e_{mm} \otimes X_{m-1}.
\]
As a consequence, one has:
\[
(\tau \boxtimes \pi_0)|_H = \tau|_{E_{m-1}} \boxtimes \pi_0.
\]
We now note the following well-known proposition due to Gelfand and Kazhdan.

**Proposition 21.2.** Let \(\tau\) be a supercuspidal representation of \(\text{GL}_m(k)\) and consider its restriction to the mirabolic subgroup \(E_{m-1}\). Then
\[
\tau|_{E_{m-1}} \cong \text{ind}^{E_{m-1}}_{U_{m}} \psi
\]
where \(U_m\) is the maximal unipotent subgroup (of upper triangular unipotent matrices) in \(E_{m-1}\) and \(\psi\) is a generic character on \(U_m\).

By induction in stages, it follows from the proposition that
\[
\text{ind}^{\text{SO}(W)}_H (\tau \otimes \pi_0)|_H \cong \text{ind}^{\text{SO}(W)}_{B_{Y,X_{m-1}}} \pi_0 \otimes \psi_t.
\]
Thus, by dualizing and Frobenius reciprocity, one has
\[
\text{Hom}_{\text{SO}(W)}(\Pi(\tau, \pi_0), \pi_V) \cong \text{Hom}_{B_{Y,X_{m-1}}} (\pi, \pi_V \otimes \overline{\psi_t}).
\]
This completes the proof of the theorem.

We note two corollaries of the theorem and [AGRS], both of which follow by applying the theorem for appropriate choices of the supercuspidal representation \(\tau\).

**Corollary 21.3.** The multiplicity 1 theorem holds for Bessel models of orthogonal and unitary groups. In other words, for any irreducible representations \(\pi\) and \(\pi_0\) of \(\text{SO}(V)\) and \(\text{SO}(W)\) respectively,
\[
\dim \text{Hom}_{U(W),N}(\pi, \pi_0 \otimes \psi_t) \leq 1.
\]

**Corollary 21.4.** Conjecture 6.1 implies Conjecture 20.2(2) for tempered representations.
In Theorem 21.1, we have calculated the branching from \( \text{SO}(V) \) to \( \text{SO}(W) \) assuming that the representation of the larger group \( \text{SO}(V) \) is a principal series representation. One can similarly calculate the branching from \( \text{SO}(V) \) to \( \text{SO}(W) \) assuming that the representation of the smaller group \( \text{SO}(W) \) is a principal series representation. We merely state the end result.

**Theorem 21.5.** Let \( W \subset V \) be a non-degenerate quadratic subspace of codimension 1 over a non-Archimedean local field \( k \). Suppose that
\[
W = Y_m \oplus W_0 \oplus Y_m^\vee
\]
and
\[
V = Y_m \oplus V_0 \oplus Y_m^\vee,
\]
with \( Y_m \) and \( Y_m^\vee \) isotropic subspaces and \( W_0 \subset V_0 \). Let \( P_W(Y_m) \) be the parabolic in \( \text{SO}(W) \) stabilizing \( Y_m \) with Levi subgroup
\[
M = \text{GL}(Y_m) \times \text{SO}(W_0)
\]
For an irreducible supercuspidal representation \( \tau \) of \( \text{GL}(Y_m) \) and an irreducible admissible representation \( \pi_0 \) of \( \text{SO}(W_0) \), let
\[
I(\tau, \pi_0) = \text{Ind}_{P(Y_m)}^{\text{SO}(W)} \tau \boxtimes \pi_0
\]
be the corresponding (unnormalized) principal series representation of \( \text{SO}(W) \). Let \( \pi \) be an irreducible admissible representation \( \text{SO}(V) \) which does not belong to the Bernstein component associated to \((\text{GL}(Y_m) \times \text{SO}(V_0), \tau \boxtimes \mu)\) for any irreducible representation \( \mu \) of \( \text{SO}(V_0) \). Then
\[
B_0(\pi, I(\tau, \pi_0)) \cong B_m(\pi, \pi_0).
\]
In other words,
\[
\text{Hom}_{\text{SO}(W)}(\pi \otimes I(\tau, \pi_0), \mathbb{C}) \cong \text{Hom}_{\text{SO}(W_0), N}(\pi \otimes (\pi_0 \otimes \psi_\ell), \mathbb{C}).
\]

22. **Fourier-Jacobi Models for Symplectic Groups**

The Bessel model conjecture addresses the restriction of representations from \( U(V) \) to \( U(W) \) when \( W \) is a subspace of \( V \) of odd codimension. One may ask for an analogous restriction problem when \( W \) has even codimension in \( V \). Such a restriction problem involves the so-called Fourier-Jacobi models. The natural setting for the consideration of Fourier-Jacobi models is the setting of symplectic groups. Because of the lack of a suitable reference in the literature, we shall discuss the case of symplectic groups in this section.

Let \( W_0 \) be a non-degenerate symplectic subspace of a symplectic space \( W \). Write
\[
W = X \oplus W_0 \oplus X^\vee
\]
where $X$ and $X^\vee$ are isotropic subspaces of $W$ which are in duality with each other. Let $P(X) = M(X) \cdot N(X)$ be the parabolic subgroup in $\text{Sp}(W)$ stabilizing the subspace $X$, with Levi subgroup

$$M(X) \cong \text{GL}(X) \times \text{Sp}(W_0)$$

stabilizing both $X$ and $X^\vee$, and unipotent radical $N(X)$ sitting in an exact sequence of $M$-modules,

$$0 \to \text{Sym}^2 X \to N(X) \to X \otimes W_0 \to 0.$$

The unipotent group $N(X)$ is thus a 2-step nilpotent group with $\text{Sym}^2(X)$ in its center. The commutator map

$$[-,-] : N(X) \times N(X) \to N(X)$$

gives rise to a skew-symmetric bilinear form

$$\Lambda^2(X \otimes W_0) \to \text{Sym}^2(X),$$

or equivalently by duality,

$$\text{Sym}^2(X^\vee) \to \Lambda^2(X^\vee \otimes W_0).$$

In our case, this last map is the reflection of the fact that, using the symplectic structure on $W_0$, symmetric bilinear forms on $X$ can be naturally embedded in the space of skew-symmetric bilinear forms on $X \otimes W_0$.

Let

$$\ell : X \to \mathbb{G}_a$$

be a nonzero homomorphism. This gives rise to a linear map

$$\text{Sym}^2(\ell) : \text{Sym}^2(X) \to \mathbb{G}_a,$$

as well as a map $\ell : X \otimes W_0 \to W_0$. As a consequence, one has the following commutative diagram of groups, which realizes the Heisenberg group $H(W_0)$ as a quotient of $N(X)$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Sym}^2(X) & \longrightarrow & N(X) & \longrightarrow & X \otimes W_0 & \longrightarrow & 0 \\
\text{Sym}^2(\ell) \downarrow & & \downarrow & & \ell \downarrow & & & & \\
0 & \longrightarrow & \mathbb{G}_a & \longrightarrow & H(W_0) & \longrightarrow & W_0 & \longrightarrow & 0.
\end{array}
$$

Therefore, given a nontrivial character $\psi : k \to \mathbb{C}^\times$, one may consider the unique irreducible representation $\omega_{W_0,\psi}$ of $H(W_0)$ with central character $\psi$ and pulling back by the above diagram, one obtains an irreducible representation $\omega_{W_0,\psi\ell}$ of $N(X)$ with central character

$$\psi_\ell = \psi \circ \text{Sym}^2(\ell).$$
One knows that this representation of \( N(X) \) can be extended to an irreducible representation of \( \widehat{\text{Sp}}(W_0) \cdot N(X) \), where \( \widehat{\text{Sp}}(W_0) \) is the unique two fold cover of the symplectic group \( \text{Sp}(W_0) \) (usually called the metaplectic group). The representation

\[
\omega_{W_0,\psi,\ell} \text{ of } \widehat{\text{Sp}}(W_0) \cdot N(X)
\]

is called a Weil representation. This Weil representation depends on the central character \( \psi_\ell \), up to the action of \( k^\times \). Thus we see that there are \( \#(k^\times / k^\times 2) \) such Weil representations.

Now note that the stabilizer in \( \text{GL}(X) \) of \( \ell \) is a mirabolic subgroup \( \text{GL}(X)_\ell \). Let \( U_X \subset \text{GL}(X)_\ell \) be a maximal unipotent subgroup. Then we define the Jacobi group associated to \( W_0 \) and \( X \) to be

\[
J_{W_0,X} = (U_X \times \widehat{\text{Sp}}(W_0) \ltimes N(X))
\]

The group \( J_{W_0,X} \) depends only on the subspace \( W_0 \subset W \) up to conjugacy in \( \text{Sp}(W) \). If \( \psi_X \) be a generic character of \( U_X \), then we have the representation

\[
\psi_X \boxtimes \omega_{W_0,\psi_\ell} \text{ of } J_{W_0,X}.
\]

As we remarked above, the representation \( \omega_{W_0,\psi_\ell} \) depends on the central character \( \psi_\ell \) up to the action of \( k^\times \). Without loss of generality, we shall fix the homomorphism \( \ell \) and allow \( \psi \) to vary. Thus, we shall suppress \( \ell \) from the notation henceforth. For example, we shall simply write \( \omega_{W_0,\psi} \).

We now come to the notion of Fourier-Jacobi model.

**Definition (Fourier-Jacobi model):** Consider \( W = X \oplus W_0 \oplus X^\vee \) as above, where \( W_0 \) has codimension \( 2d \) in \( W \). Fix an additive character \( \psi \) of \( k \). Let \( \pi \) and \( \pi_0 \) be irreducible admissible representations of \( \text{Sp}(W) \) and \( \widehat{\text{Sp}}(W_0) \) respectively. The \( (\pi_0, \psi) \)-Fourier-Jacobi model of \( \pi \) is the space

\[
FJ_{\psi,d}(\pi, \pi_0) = \text{Hom}_{J_{W_0,X}}(\pi \otimes (\psi_X \boxtimes (\pi_0 \otimes \omega_{W_0,\psi})), \mathbb{C}).
\]

Observe that, in the above definition, the representation \( \pi_0 \) is regarded as a representation of \( \widehat{\text{Sp}}(W_0) \cdot H(W_0) \) through the natural map \( \widehat{\text{Sp}}(W_0) \cdot H(W_0) \to \widehat{\text{Sp}}(W_0) \), so that \( \pi_0 \otimes \omega_{W_0,\psi} \) is actually a representation of \( \text{Sp}(W_0) \cdot N(X) \). Similarly, one has the analogous definition if \( \pi \) is a representation of \( \widehat{Sp}(W) \) and \( \pi_0 \) is a representation of \( \text{Sp}(W_0) \).

Note that the above definition makes sense even if \( d = 0 \). In that case, \( X = 0 \) and \( W_0 = W \), so that

\[
FJ_{\psi,0}(\pi, \pi_0) = \text{Hom}_{\text{Sp}(W)}(\pi \otimes \pi_0 \otimes \omega_{W,\psi}, \mathbb{C}).
\]

This should be considered as the analog of the restriction problem from \( \text{SO}(n) \) to \( \text{SO}(n-1) \).
In Section 24, we shall state a conjecture concerning the non-vanishing of the spaces $F_{J_{\psi,d}}(\pi, \pi_0)$. This conjecture is analogous to the Bessel model conjecture in the orthogonal case [GP2]. Not surprisingly, we shall then show that the conjecture for $F_{J_{\psi,d}}$ follows from the case $d = 0$. This is a consequence of Theorem 22.1 below.

Before coming to Theorem 22.1, we recall that a parabolic subgroup $\hat{P}$ in $\hat{Sp}(W)$ is nothing but the inverse image of a parabolic $P$ in $Sp(W)$. It is known that the metaplectic covering splits (uniquely) over unipotent subgroups, so for a Levi decomposition $P = M \cdot N$, it makes sense to talk of the corresponding Levi decomposition $\hat{P} = \hat{M} \cdot N$ in $\hat{Sp}(W)$. Furthermore, we note that for a Levi subgroup of the form $M = GL(X) \cdot Sp(W_0)$ in $Sp(W)$,

$$\hat{M} = \left(\hat{GL}(X) \times \hat{Sp}(W_0)\right) / \Delta \mu_2$$

where $\hat{GL}(X)$ is a certain two fold cover of $GL(X)$ defined as follows. As a set, we write

$$\hat{GL}(X) = GL(X) \times \{\pm 1\},$$

and the multiplication is given by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot (\det g_1, \det g_2)_2),$$

where $(-, -)_2$ denotes the Hilbert symbol on $k^\times$ with values in $\{-1, 1\}$.

The two fold cover $\hat{GL}(X)$ has a natural genuine 1-dimensional character

$$\chi_\psi : \hat{GL}(X) \longrightarrow \mathbb{C}^\times$$

defined as follows. The determinant map gives rise to a natural group homomorphism

$$\det : \hat{GL}(X) \longrightarrow \hat{GL}(\wedge^{top}X) = \hat{GL}(1).$$

On the other hand, one has a genuine character on $\hat{GL}(1)$ defined by

$$(a, \epsilon) \mapsto \epsilon \cdot \gamma(a, \psi)^{-1},$$

where

$$\gamma(a, \psi) = \gamma(\psi_a)/\gamma(\psi)$$

and $\gamma(\psi)$ is an 8-th root of unity associated to $\psi$ by Weil. Composing this character with $\det$ gives the desired genuine character $\chi_\psi$ on $\hat{GL}(X)$, which satisfies:

$$\chi_\psi^2(g, \epsilon) = (\det(g), -1)_2.$$
Note that this bijection depends on the additive character $\psi$ of $k$. Now associated to a representation $\tau$ of $\text{GL}(X)$ and $\pi_0$ of $\hat{\text{Sp}}(W_0)$, one has the representation $$\tilde{\tau}_\psi \boxtimes \pi_0$$ of $\hat{M}$.

Then one can consider the (unnormalized) induced representation $$I_\psi(\tau, \pi_0) = \text{Ind}_{\hat{P}}^{\hat{M}}((\tilde{\tau}_\psi \otimes \pi_0)).$$

Here is the main result of this section, which reduces the computation of $F_{J, \psi, d}$ for general $d$ to the case $d = 0$. It is the analog of Theorem 21.1.

**Theorem 22.1.** Consider $W = X \oplus W_0 \oplus X^\vee$ as above with $\dim X = d$ and fix the additive character $\psi$. Let

- $\tau$ be a supercuspidal representation of $\text{GL}(X)$;
- $\pi_0$ be a genuine representation of $\hat{\text{Sp}}(W_0)$;
- $\pi$ be an irreducible representation of $\text{Sp}(W)$,

and consider the (unnormalized) induced representation $I_\psi(\tau, \pi_0)$ of $\text{Sp}(W)$. Assume that $\pi^\vee$ does not belong to the Bernstein component associated to $(\text{GL}(X) \times \text{Sp}(W_0), \tau \boxtimes \mu)$ for any representation $\mu$ of $\text{Sp}(W_0)$. Then

$$F_{J, \psi, 0}(I_\psi(\tau, \pi_0), \pi) \cong F_{J, \psi, d}(\pi, \pi_0).$$

In other words,

$$\text{Hom}_{\text{Sp}(W)}(I_\psi(\tau, \pi_0) \otimes \pi \otimes \omega_{W, \psi}, \mathbb{C}) \cong \text{Hom}_{Jw_0, X}(\pi \otimes (\psi_X \boxtimes (\pi_0 \otimes \omega_{W_0, \psi})), \mathbb{C}).$$

**Proof.** Let $P = P(X) = M(X) \cdot N(X)$ be the parabolic subgroup in $\text{Sp}(W)$ stabilizing the subspace $X$. Recall that

$$\hat{M}(X) \cong \hat{\text{GL}}(X) \times \hat{\text{Sp}}(W_0))/\Delta \mu_2$$

and we have fixed an element

$$\ell \in \text{Hom}(X, G_a) \cong X^\vee$$

which leads to a quotient map

$$N(X) \rightarrow H(W_0).$$

Now the Weil representation $\omega_{W, \psi}$ has a convenient description as a $\hat{P}$-module; this is the so-called mixed model of the Weil representation. This model of $\omega_{W, \psi}$ is realized on the space

$$S(X^\vee) \otimes \omega_{W_0, \psi}$$

of Schwarz-Bruhat functions on $X^\vee$ valued in $\omega_{W_0, \psi}$. In particular, evaluation at 0 gives a $\hat{P}$-equivariant map

$$ev : \omega_{W, \psi} \rightarrow \chi_{\psi} |\det_X|^{1/2} \boxtimes \omega_{W_0, \psi},$$
where \( N(X) \) acts trivially on the target space. In fact, this map is the projection of \( \omega_{W,\psi} \) onto its space of \( N(X) \)-coinvariants. From this, one deduces the following short exact sequence of \( \widehat{P} \)-modules:

\[
0 \longrightarrow \text{ind}_{\widehat{P}}^\widehat{P} \chi_\psi |\det_X|^{1/2} \otimes \omega_{W_0,\psi} \longrightarrow \omega_{W,\psi} \longrightarrow \chi_\psi |\det_X|^{1/2} \otimes \omega_{W_0,\psi} \longrightarrow 0,
\]

where \( \widehat{P}_\ell = \left( (\widehat{GL}(X) \times \widehat{Sp}(W_0))/\Delta \mu_2 \right) \cdot N(X) \) is the stabilizer of \( \ell \) in \( \widehat{P} \) and its action on \( \omega_{W_0,\psi} \) is via the Weil representation of \( \widehat{Sp}(W_0) \cdot N(X) \). Moreover, the compact induction functor \( \text{ind} \) is unnormalized.

Tensoring the above short exact sequence by \( \tilde{\tau}_\psi \otimes \pi_0 \) and then inducing to \( \widehat{Sp}(W) \), one gets a short exact sequence of \( Sp(W) \)-modules:

\[
0 \longrightarrow \text{ind}_{\widehat{P}}^{\widehat{Sp}(W)} |\det_X|^{1/2} \cdot \chi_\psi^2 \cdot \tau |_{GL(X)_\ell} \otimes (\pi_0 \otimes \omega_{W_0,\psi}) = A \leftarrow \text{ind}_{\widehat{P}}^{\widehat{Sp}(W)} |\det_X|^{1/2} \otimes (\pi_0 \otimes \omega_{W_0,\psi}) = C \leftarrow 0.
\]

By our assumption on \( \pi \),

\[
\text{Hom}_{\widehat{Sp}(W)}(C, \pi^\vee) = 0 \quad \text{and} \quad \text{Hom}_{\widehat{Sp}(W)}(B, \pi^\vee) = \text{Hom}_{\widehat{Sp}(W)}(A, \pi^\vee).
\]

Moreover, by Proposition 21.2,

\[
\tau |_{GL(X)_\ell} \cong \text{ind}_{\ell X}^{GL(X)_\ell} \psi_X,
\]

so that

\[
A = \text{ind}_{W_0, X}^{\widehat{Sp}(W)} (\psi_X \otimes \pi_0) \otimes \omega_{W_0,\psi}.
\]

Therefore, the desired result follows by Frobenius reciprocity.
23. LOCAL LANGLANDS CONJECTURE FOR $\widehat{Sp}(2n)$

In this section, we shall discuss the local Langlands correspondence for the metaplectic groups $\widehat{Sp}(W)$. This is necessary for the statement of the conjecture for Fourier-Jacobi models given in the next section.

Though $\widehat{Sp}(W)$ is not a linear group, it is a folklore that were it to have a Langlands dual group at all, the only possible candidate is the symplectic group $Sp_{2n}(C)$ (where $2n = \dim W$). This expectation is justified for the group $\widehat{Sp}(2)$ by the fundamental work of Waldspurger [W], and over $R$ by the work of Adams-Barbasch [AB]. In fact, one has the following theorem of Kudla-Rallis [KR]:

**Theorem 23.1. (Kudla-Rallis)** Suppose that the residue characteristic of $k$ is odd. Relative to the choice of an additive character $\psi$ of $k$, there is a natural bijection between

$$\Pi(\widehat{Sp}(2n)) := \{\text{irreducible genuine representations of } \widehat{Sp}(2n,k)\}$$

and the disjoint union

$$\Pi(SO(2n + 1)) \cup \Pi(SO^*(2n + 1))$$

of the set of irreducible representations of the split $SO(2n + 1,k)$ and the non-split $SO^*(2n + 1,k)$.

Before giving a sketch of the proof of this theorem, we deduce the following corollary:

**Corollary 23.2.** Assume that the residue characteristic of $k$ is odd. Suppose that the local Langlands conjecture as formulated in Section 4 holds for $SO(2n + 1)$ and $SO^*(2n + 1)$. Then one has a bijection (depending on $\psi$)

$$\Pi(\widehat{Sp}(2n)) \leftrightarrow \Phi(\widehat{Sp}(2n)),$$

where $\Phi(\widehat{Sp}(2n))$ is the set of conjugacy classes of pairs $(\varphi, \rho)$ such that

$$\varphi : W^\varphi_k \longrightarrow Sp_{2n}(C)$$

is a Langlands parameter and $\rho$ is an irreducible character of the component group $A_{\varphi}$.

**Proof.** The reader will not find Theorem 23.1 in the reference [KR], so let us explain how it follows from the results there. The natural bijection of the Theorem is given by the theta correspondence between the dual pairs

$$\widehat{Sp}(2n) \times \begin{cases} O(2n + 1); \\ O^*(2n + 1). \end{cases}$$

This theta correspondence depends on the choice of $\psi$, but since we will be fixing $\psi$, we suppress it from the notation.
Given an irreducible representation $\sigma$ of $\widehat{Sp}(2n)$, we shall let $\Theta(\sigma)$ (respectively $\Theta^*(\sigma)$) denote the big theta lift of $\sigma$ to $O(2n+1)$ (resp. $O^*(2n+1)$), and we let $\theta(\sigma)$ (resp. $\theta^*(\sigma)$) be the maximal semisimple quotient of $\Theta(\sigma)$ (resp. $\Theta^*(\sigma)$). Similarly, starting with a representation $\pi$ of $O(2n+1)$ or $O^*(2n+1)$, one has the representations $\Theta(\pi)$ and $\theta(\pi)$ of $\widehat{Sp}(2n)$.

Since the residue characteristic of $k$ is odd, one knows by a result of Waldspurger (proving the so-called Howe’s conjecture) that the representations $\theta(\sigma)$, $\theta^*(\sigma)$ and $\theta(\pi)$ are irreducible or zero. This is the only place where we use the assumption on the residue characteristic of $k$.

We now divide the proof into two steps:

(i) Given an irreducible representation $\sigma$ of $\widehat{Sp}(2n)$, exactly one of $\Theta(\sigma)$ or $\Theta^*(\sigma)$ is nonzero.

Indeed, [KR, Thm. 3.8] shows that any irreducible representation $\sigma$ of $\widehat{Sp}(2n)$ participates in theta correspondence with at most one of $O(2n+1)$ or $O^*(2n+1)$. We claim however that $\sigma$ does have nonzero theta lift to $O(2n+1)$ or $O^*(2n+1)$. To see this, note that [KR, Prop. 4.1] shows that $\sigma$ has nonzero theta lift to one of $O(2n+1)$ or $O^*(2n+1)$ if and only if $\text{Hom}_{\text{c}Sp(2n) \times \text{c}Sp(2n)}(I_P(0), \sigma \boxtimes \sigma^\vee) \neq 0$. Here, $I_P(s)$ denotes the degenerate principal series representation of $\widehat{Sp}(4n)$ unitarily induced from the character $\chi_\psi \cdot |\det|^s$ of the Siegel parabolic subgroup. We thus need to show that this Hom space is nonzero. This can be achieved by the doubling method of Piatetski-Shapiro and Rallis, which provides a zeta integral

$$Z(s) : I_P(s) \otimes \sigma \otimes \sigma^\vee \to \mathbb{C}.$$ 

The precise definition of $Z(s)$ need not concern us here; it suffices to note that for a flat section $\Phi(s) \in I_P(s)$ and $f \otimes f^\vee \in \sigma \otimes \sigma^\vee$, $Z(s, \Phi(s), f \otimes f^\vee)$ is a meromorphic function in $s$. Moreover, at any $s = s_0$, the leading term of the Laurent expansion of $Z(s)$ gives a nonzero element

$$Z^*(s_0) \in \text{Hom}_{\text{c}\widehat{Sp}(2n) \times \text{c}\widehat{Sp}(2n)}(I_P(s_0), \sigma \boxtimes \sigma^\vee).$$

This proves our contention that $\sigma$ participates in the theta correspondence with exactly one of $O(2n+1)$ or $O^*(2n+1)$.

By (i), one obtains a map

$$\Pi(\widehat{Sp}) \to \Pi(O(2n+1)) \cup \Pi(O^*(2n+1)).$$

Moreover, this map is injective by the result of Waldspurger alluded to above.

(ii) An irreducible representation $\pi_0$ of $SO(2n+1)$ has two extensions to $O(2n+1) = SO(2n+1) \times \langle \pm 1 \rangle$, and exactly one of these extensions participates in the theta correspondence with $\widehat{Sp}(2n)$.
Suppose on the contrary that $\pi$ is an irreducible representation of $O(2n + 1)$ such that both $\pi$ and $\pi \otimes \det$ participates in theta correspondence with $\widehat{Sp}(2n)$, say $\sigma = \theta(\pi)$ and $\sigma' = \theta(\pi \otimes \det)$.

Now consider the seesaw diagram:

\[
\begin{array}{c}
\widehat{Sp}(4n) \\
\downarrow \\
\widehat{Sp}(2n) \times \widehat{Sp}(2n) \\
\downarrow \\
O(2n + 1)
\end{array}
\begin{array}{c}
O(2n + 1) \times O(2n + 1) \\
\downarrow \\
\widehat{Sp}(2n) \\
\downarrow \\
O(2n + 1)
\end{array}
\]

The seesaw identity implies that
\[
\text{Hom}_{\widehat{Sp}(2n) \times \widehat{Sp}(2n)}(\Theta(\det), \sigma \boxtimes \sigma') \supset \text{Hom}_{O(2n+1)}((\pi \otimes \det) \otimes \pi^\vee, \det) \neq 0.
\]

This implies that $\Theta(\det) \neq 0$. However, a classical result of Rallis says that the determinant character of $O(2n + 1)$ does not participate in the theta correspondence with $\widehat{Sp}(2r)$ for $r \leq n$. This gives the desired contradiction.

We have thus shown that at most one of $\pi$ or $\pi \otimes \det$ could have nonzero theta lift to $\widehat{Sp}(2n)$. On the other hand, the analog of the zeta integral argument in (i) shows that one of $\pi$ or $\pi \otimes \det$ does lift to $\widehat{Sp}(2n)$. This proves (ii).

Putting (i) and (ii) together, we see that the composite map
\[
\Pi(\widehat{Sp}(2n)) \longrightarrow \Pi(O(2n + 1)) \cup \Pi(O^*(2n + 1)) \longrightarrow \Pi(SO(2n + 1)) \cup \Pi(SO^*(2n + 1))
\]
provided by theta correspondence is bijective, as desired.

As we remarked in the proof of the theorem, the only reason for the assumption of odd residue characteristic is that Howe’s conjecture for local theta correspondence is only known under this assumption. In the rest of the paper, we shall assume that Howe’s conjecture is known for even residue characteristic as well, so that the results of Theorem 23.1 can be applied.

We conclude this section with a brief description of the local Langlands conjecture for $Sp(2m)$. The dual group of $Sp(2m)$ is the group $SO_{2m+1}(\mathbb{C})$, which is adjoint. Thus, one expects that the set $\Pi(\widehat{Sp}(2m))$ of irreducible representations of $Sp(2n)$ is in bijection with the set of conjugacy classes of pairs $(\varphi, \rho)$ where
\[
\varphi : W'_k \longrightarrow SO_{2m+1}(\mathbb{C})
\]
is a Langlands parameter and $\rho$ is an irreducible character of the component group $A_\varphi$.

Putting the above discussions together, we see that a Langlands parameter for the group $\text{Sp}(2m) \times \widehat{\text{Sp}}(2n)$ is a homomorphism

$$\varphi : W'_k \longrightarrow \text{SO}_{2m+1}(\mathbb{C}) \times \text{Sp}_{2n}(\mathbb{C}).$$

Since the group $\text{SO}_{2m+1}(\mathbb{C}) \times \text{Sp}_{2n}(\mathbb{C})$ has a natural symplectic representation given by the tensor product, we may regard $\varphi$ as a symplectic representation of dimension $2n \cdot (2m + 1)$. It is this symplectic representation (which depends on $\psi$) which controls the behavior of Fourier-Jacobi models for symplectic and metaplectic groups.

24. CONJECTURES FOR FOURIER-JACOBI MODELS

In this section, we will give a conjecture for the non-vanishing of the Fourier-Jacobi models $FJ_{\psi,k}$, which is analogous to the Bessel model conjecture given earlier. We shall make use of the Langlands correspondence for $\widehat{\text{Sp}}(2n)$ and $\text{Sp}(2n)$ described in the previous section.

Conjecture 24.1. Suppose that $W_0$ and $W$ are symplectic spaces such that $|\dim W - \dim W_0| = 2d$.

1. For any irreducible (genuine) representations $\pi$ and $\pi_0$ of $\text{Sp}(W)$ and $\widehat{\text{Sp}}(W_0)$, $\dim FJ_{\psi,d}(\pi, \pi_0) \leq 1$

2. Let $\varphi$ be a generic Langlands parameter for $\text{Sp}(W) \times \widehat{\text{Sp}}(W_0)$. Then $\sum_{\pi \otimes \pi_0 \in \Pi(\varphi)} \dim FJ_{\psi,d}(\pi, \pi_0) = 1$.

3. The unique representation $\pi \otimes \pi_0$ which has nonzero contribution to the sum in (2) is given by a character on the component group $A_\varphi$, specified by the same recipe as in [GP1, Conj].

In fact, Theorem 22.1 implies:

**Proposition 24.2.** Conjecture 24.1 for general $d$ follows from the case $d = 0$.

We shall now focus on the Fourier-Jacobi conjecture for $FJ_{\psi,0}$. In particular, we shall see that the Fourier-Jacobi conjecture for $FJ_{\psi,0}$ is related to the Bessel conjecture for $B_0$, i.e. the restriction problem from $\text{SO}(n)$ to $\text{SO}(n - 1)$. This link is again provided by the theta correspondence.

More precisely, consider the see-saw diagram:
Let $\pi$ be an irreducible representation of $SO(2n+1)$ and $\sigma$ an irreducible representation of $\text{Sp}(2n)$. Then the local see-saw identity says that

\[ \dim \text{Hom}_{\text{Sp}(2n)}(\Theta_{\psi}(\pi) \otimes \omega_{\psi}, \sigma) = \dim \text{Hom}_{SO(2n+1)}(\Theta_{\psi}(\sigma), \pi), \]

where $\Theta_{\psi}(\pi)$ denotes the big theta lift of $\pi$. As a result of this identity, and the multiplicity one theorems of [AGRS], we have:

**Theorem 24.3.** (i) Suppose that $\sigma$ is a discrete series representation of $\text{Sp}(2n)$. Then for any irreducible representation $\sigma_0$ of $\hat{\text{Sp}}(2n)$, we have

\[ \dim \text{Hom}_{\text{Sp}(2n)}(\sigma \otimes \sigma_0 \otimes \omega_{\psi}, \mathbb{C}) \leq 1. \]

(ii) Suppose further that $\pi$ is a discrete series representation of $SO(2n+1)$. Then

\[ \text{Hom}_{\text{Sp}(2n)}(\sigma^\vee \otimes \theta(\pi) \otimes \omega_{\psi}^\vee, \mathbb{C}) \cong \text{Hom}_{SO(2n+1)}(\theta_{\psi}(\sigma) \otimes \pi^\vee, \mathbb{C}). \]

**Proof.** This follows by the see-saw identity, the multiplicity one theorems of [AGRS] and a result of Muic which says that, for the dual pairs under consideration, $\Theta_{\psi}(\pi) = \theta_{\psi}(\pi)$ for discrete series representations $\pi$. \hfill $\square$

It follows from the theorem that, modulo issues of $\Theta_{\psi}$ versus $\theta_{\psi}$ and assuming that the local theta correspondence is described in terms of the Langlands parameter in the expected way, the conjectures for $B_0$ and $FJ_{\psi,0}$ are essentially equivalent.

25. **Fourier-Jacobi Models for Unitary Groups**

In this section, we discuss the conjecture about Fourier-Jacobi models of unitary groups.

Let $K$ be a quadratic extension of a field $k$ and let $W$ be a vector space over $K$, equipped with a non-degenerate skew-Hermitian form:

\[ \langle -, - \rangle : W \times W \rightarrow K. \]

The automorphism group $U(W)$ of the skew-Hermitian space is a unitary group. Indeed, if $\delta \in K^\times$ is a trace zero element, then multiplication by $\delta$ gives a bijection between skew-Hermitian forms and Hermitian forms on $W$. 
Let $W_0$ be a non-degenerate subspace of $W$ with
\[ W = X \oplus W_0 \oplus X^\vee \]
where $X$ and $X^\vee$ are isotropic subspaces of $W$ in duality with each other. Let $P(X)$ be the parabolic subgroup in $U(W)$ stabilizing the subspace $X$, and $M(X)$ the Levi subgroup of $P(X)$ which stabilizes both $X$ and $X^\vee$, so that
\[ M(X) = \text{GL}_K(X) \times U(W_0). \]
The unipotent radical $N(X)$ of $P(X)$ sits in an exact sequence of $M(X)$-modules,
\[ 0 \rightarrow \text{Sym}^2(X, \epsilon) \rightarrow N(X) \rightarrow X \otimes W_0 \rightarrow 0, \]
where $\text{Sym}^2(X, \epsilon)$ denotes the space of Hermitian 2-forms on $X^\vee$. Thus, $N(X)$ is a 2-step nilpotent group with $\text{Sym}^2(X, \epsilon)$ as its center. The commutator map
\[ [-, -] : N(X) \times N(X) \rightarrow N(X) \]
gives rise to a linear map
\[ \Lambda^2_k(X \otimes W_0) \rightarrow \text{Sym}^2(X, \epsilon) \]
where $\Lambda^2_k(X \otimes W_0)$ denotes 2nd exterior power of $X \otimes_K W_0$ considered as a vector space over $k$. By duality, we get a $k$-linear map
\[ \text{Sym}^2(X^\vee, \epsilon) \rightarrow \Lambda^2(X^\vee \otimes W_0). \]
This last map is a reflection of the fact that, since $W_0$ has a skew-Hermitian structure, Hermitian forms on $X$ can be embedded in the space of skew-symmetric $k$-bilinear forms on $X \otimes W_0$ (by taking the $K/k$-trace of the form on the tensor product).

Let
\[ \ell : X \rightarrow K \]
be a nonzero $K$-linear homomorphism. This gives rise to a $k$-linear map
\[ \text{Sym}^2(\ell) : \text{Sym}^2(X, \epsilon) \rightarrow k \]
by the recipe
\[ \text{Sym}^2(\ell)(v \otimes v) = \ell(v)\overline{\ell(v)} \text{ for } v \in X. \]
The map $\ell : X \rightarrow K$ also gives rise to a map
\[ W_0 \otimes X \rightarrow W_0, \]
and one obtains the following commutative diagram of groups, which realizes the Heisenberg group $H(W_0)$ as a quotient of $N(X)$:
\[ \begin{array}{c}
0 \rightarrow \text{Sym}^2(X, \epsilon) \rightarrow N(X) \rightarrow X \otimes W_0 \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow k \rightarrow H(W_0) \rightarrow W_0 \rightarrow 0.
\end{array} \]
Here, we are regarding $W_0$ as a $k$-vector space with a non-degenerate symplectic form.
inherited from the given skew-Hermitian form (by taking the trace from $K$ to $k$ of the skew-Hermitian form); this gives us an embedding

\[ U(W_0) \hookrightarrow \text{Sp}(W_0). \]

Fix a non-trivial additive character $\psi$ of $k$. As before, we have the unique irreducible representation of $H(W_0)$ with central character $\psi$, and pulling this back to $N(X)$ by the above diagram, we obtain an irreducible representation of $N(X)$ with central character

\[ \psi_\ell = \psi \circ \text{Sym}^2(\ell). \]

Further, one knows that this representation of $N(X)$ can be extended to an irreducible representation of $U(W_0) \ltimes N(X)$. This extension is, however, non-unique since $U(W_0)$ is not its own commutator. In fact, the choice of a character $\mu$ of $K^\times$ for which $\mu |_{k^\times} = \omega_{K/k}$ determines such an extension. Therefore we have an irreducible representation

\[ \omega_{W_0,\psi,\ell,\mu} \text{ of } U(W_0) \ltimes N(X), \]

which we call a Weil representation.

Now note that the stabilizer in $\text{GL}(X)$ of $\ell$ is a mirabolic subgroup $\text{GL}(X)_\ell$. Let $U_X \subset \text{GL}(X)_\ell$ be a maximal unipotent subgroup. Then we define the *Jacobi group* associated to $W_0$ and $X$ by

\[ J_{W_0,X} = (U_X \times U(W_0)) \ltimes N(X). \]

The group $J_{W_0,X}$ depends only on the subspace $W_0 \subset W$ up to conjugacy in $U(W)$. If $\psi_X$ is a generic character of $U_X$, then we have the representation

\[ \psi_X \boxtimes \omega_{W_0,\psi,\ell,\mu} \text{ of } J_{W_0,X}. \]

We now come to the notion of Fourier-Jacobi model in the unitary setting.:

**Definition (Fourier-Jacobi model):** Consider $W = X \oplus W_0 \oplus X^\vee$ as above, where $W_0$ has codimension $2d$ in $W$. Fix an additive character $\psi$ of $k$ and a character $\mu$ of $K^\times$ with $\mu |_{k^\times} = \omega_{K/k}$. Let $\pi$ and $\pi_0$ be irreducible representations of $U(W)$ and $U(W_0)$ respectively. Then the $(\pi_0, \psi, \mu)$-Fourier-Jacobi model of $\pi$ is the space

\[ FJ_{\psi,\mu,d}(\pi, \pi_0) = \text{Hom}_{J_{W_0,X}}(\pi \otimes (\psi_X \boxtimes (\pi_0 \otimes \omega_{W_0,\psi,\mu})), \mathbb{C}). \]

Note that when $d = 0$, so that $X = 0$ and $W_0 = W$, we have

\[ FJ_{\psi,\mu,0}(\pi, \pi_0) = \text{Hom}_{U(W)}(\pi \otimes \pi_0 \otimes \omega_{W,\psi,\mu}, \mathbb{C}). \]

This should be considered the basic Fourier-Jacobi model of unitary groups.

Before stating the conjecture for Fourier-Jacobi models in the unitary case, we note the following lemma.
Lemma 25.1. The L-group of $U(m) \times U(n)$ has a natural orthogonal representation of dimension $2mn$ if $m \equiv n \mod 2$, and a symplectic representation of dimension $2mn$ if $m \not\equiv n \mod 2$.

Proof. Let $X$ be a complex vector space of dimension $m$, and let $X \oplus X^\vee$ be equipped with its canonical bilinear form which is symmetric if $m$ is odd, and skew-symmetric if $m$ is even. Similarly let $Y$ be a complex vector space of dimension $n$. The $L$-group of $U(m)$ has a natural embedding into the isometry group of this bilinear form on $X \oplus X^\vee$ leaving invariant the set $X \bigoplus X^\vee$. Given bilinear forms on $X \oplus X^\vee$ and $Y \oplus Y^\vee$, there is a natural bilinear form on the tensor product space, $(X \oplus X^\vee) \otimes (Y \oplus Y^\vee)$, in which $X \otimes Y \oplus X^\vee \otimes Y^\vee$ is a non-degenerate subspace containing $X \otimes Y$ and $X^\vee \otimes Y^\vee$ as isotropic subspaces in natural duality with each other. Clearly, $X \otimes Y \bigoplus X^\vee \otimes Y^\vee$ is left invariant under $GL(X) \times GL(Y)$, as well as for the tensor product of any automorphism of $X \otimes X^\vee$ which interchanges $X$ and $X^\vee$ with an automorphism of $Y \oplus Y^\vee$ which interchanges $Y$ and $Y^\vee$.

Clearly, if both $X \oplus X^\vee$ and $Y \oplus Y^\vee$ are quadratic spaces, or are both symplectic spaces, then $X \otimes Y \oplus X^\vee \otimes Y^\vee$ is a quadratic space, providing the natural orthogonal representation of the $L$-group of $U(m) \times U(n)$. □

Remark: The co-ordinate free approach to the $L$-group of $U(n)$ that we have used in the proof of the previous lemma, can also be used to give an embedding of $L(U(n_1) \times U(n_2) \times \cdots \times U(n_k))$ into $L(U(n_1 + n_2 + \cdots + n_k))$. This needs an introduction of a character $\mu$ of $K^\times$ such that $\mu|_{k^\times} = \omega_{K/k}$ for factors $U(n_i)$ for which the parity of $n_i$ is not the same as that of $n_1 + \cdots + n_k$ modulo 2 to flip the parity of the bilinear form on $X_i \oplus X_i^\vee$ to match that of $(X_1 \oplus \cdots \oplus X_k) \oplus (X_1 \oplus \cdots \oplus X_k)^\vee$. Note that there is a bijection between characters of $K^\times$ trivial on $k^\times$ and characters of $K^\times$ whose restriction to $k^\times$ is $\omega_{K/k}$ obtained by multiplication by this character $\mu$ of $K^\times$ which establishes an injection of the set of 2 dimensional orthogonal representations of $W_{K/k}$ into 2 dimensional symplectic representations by sending $Ind(\alpha)$ to $Ind(\alpha \mu)$ where the character $\alpha$ of $K^\times$ is trivial on $k^\times$; thus ‘multiplication by $\mu$’ has the effect of turning parameters with values in $O(X_i \oplus X_i^\vee)$ preserving $X_i \bigoplus X_i^\vee$ into parameters with values in $Sp(X_i \oplus X_i^\vee)$ preserving $X_i \bigoplus X_i^\vee$ giving rise to an embedding of $L(U(n_1) \times U(n_2) \times \cdots \times U(n_k))$ into $L(U(n_1 + n_2 + \cdots + n_k))$.

Returning now to the Fourier-Jacobi models for the unitary group, since we are in the situation of

$$\dim W \equiv \dim W_0 \mod 2,$$

if $\varphi$ is a Langlands parameter for $U(W) \times U(W_0)$ and

$$\varphi_K : W'_K \rightarrow GL(V) \times GL(V_0)$$

is its restriction to $W'_K$, then the representation $Ind(V \otimes V_0)$ is an orthogonal representation. However, as noted in the remark above, by a slight modification, we can create a symplectic representation as follows. Choose a character $\mu$ of $K^\times$ such that

$$\mu|_{k^\times} = \omega_{K/k}.$$
Then for
\[ \varphi_{K,\mu} = \varphi_K \otimes \mu \]
where we have regarded \( \mu \) as a 1-dimensional character of \( W_K \) by local class field theory,
\[ \varphi_\mu := \text{Ind}(\varphi_{K,\mu}) \]
is a symplectic representation.

Here is the conjecture about Fourier-Jacobi models of unitary groups:

**Conjecture 25.2.** Suppose that \( W_0 \subset W \) are skew-Hermitian spaces such that \( \dim W - \dim W_0 = 2d \). Fix a non-trivial additive character \( \psi \) of \( k \) and a character \( \mu \) of \( K \times K \) such that \( \mu \vert_{k \times k} = \omega_{K/k} \).

1. For any irreducible representations \( \pi \) and \( \pi_0 \) of \( U(W) \) and \( U(W_0) \),
   \[ \dim FJ_{\psi,\mu,d}(\pi,\pi_0) \leq 1 \]
2. Let \( \varphi \) be a generic Langlands parameter for \( U(W) \times U(W_0) \). Then
   \[ \sum_{\pi \otimes \pi_0 \in \Pi(\varphi)} \dim FJ_{\psi,\mu,d}(\pi,\pi_0) = 1. \]
3. The unique representation \( \pi \otimes \pi_0 \) which has nonzero contribution to the sum in (2) is given by a character on the component group \( A_{\varphi,\mu} \), specified by the same recipe as in Conjecture.

As in the previous cases, the following result reduces the computation of \( FJ_{\psi,\mu,d} \) to that of \( FJ_{\psi,\mu,0} \). Since the proof is similar to that of Theorem 22.1, we shall omit the details.

**Theorem 25.3.** Consider \( W = X \oplus W_0 \oplus X^\vee \) with \( \dim X = d \) and fix \( \psi \) and \( \mu \) as in Conjecture 25.2. Let
- \( \tau \) be a supercuspidal representation of \( \text{GL}(X) \);
- \( \pi_0 \) be an irreducible representation of \( U(W_0) \);
- \( \pi \) be an irreducible representation of \( U(W) \),
and consider the (unnormalized) induced representation
\[ I(\tau,\pi_0) = \text{Ind}^{U(W)}_{U(X)} \tau \boxtimes \pi_0. \]
Assume that \( \pi^\vee \) does not belong to the Bernstein component associated to \( (\text{GL}(X) \times U(W_0), \tau \boxtimes \mu) \) for any representation \( \mu \) of \( U(W_0) \). Then
\[ FJ_{\psi,\mu,0}(I(\tau,\pi_0),\pi) \cong FJ_{\psi,\mu,d}(\pi,\pi_0). \]
In other words,
\[
\text{Hom}_{U(W)}(I(\tau, \pi_0) \otimes \pi \otimes \omega_{W, \psi, \mu}, \mathbb{C}) \cong \text{Hom}_{J_{W_0, X}}(\pi \boxtimes (\pi_0 \otimes \omega_{W_0, \psi, \mu}), \mathbb{C}).
\]

The theorem allows one to focus on the basic Fourier-Jacobi model $F_{J, \psi, \mu, 0}$. As in the case of symplectic groups, one can relate the Fourier-Jacobi model $F_{J, \psi, \mu, 0}$ to the Bessel $B_0$, via the following seesaw diagram:

\[
\begin{array}{c}
U(n + 1) \\
\downarrow \\
U(n) \times U(1) \\
\downarrow \\
U(n)
\end{array}
\begin{array}{c}
U(n) \\
\downarrow \\
U(n) \times U(n)
\end{array}
\]

In particular, using the seesaw identity, the results of Harris-Kudla-Sweet [HST] and the result of [AGRS], one concludes:

**Theorem 25.4.** Let $\pi$ and $\pi_0$ be irreducible representations of $U(W)$ and suppose that $\pi$ is supercuspidal. Then
\[
\dim F_{J, \psi, \mu}(\pi, \pi_0) \leq 1.
\]

Ignoring issues of $\Theta$ versus $\theta$ and assuming that the local theta correspondence behaves in an expected way with respect to the local Langlands conjecture, one can make a precise relation between $F_{J, \psi, \mu, 0}$ and $B_0$. We leave the somewhat intricate details to the interested reader.

**References**


W.T.G.: Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, 92093
E-mail address: wgan@math.ucsd.edu

B.H.G: Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: gross@math.harvard.edu

D.P.: School of Mathematics, Tata Institute of Fundamental Research, Colaba, Mumbai-400005, INDIA
E-mail address: dprasad@math.tifr.res.in